Methods for representing uncertainty

A literature review

Enrico Zio and Nicola Pedroni

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Résumé

Titre Revue de littérature sur la représentation de l'incertitude dans l'analyse de risque

Mots-clefs incertitude, analyse de risque, approches possibilistes, analyse d'intervalle

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Ce document fournit une analyse critique de différentes méthodes de **représentation et d'analyse d'incertitude** dans les analyses de risques : l'analyse probabiliste classique, les probabilités imprécises (analyse d'intervalle), les fonctions de croyance de Dempster-Shafer et la théorie des possibilités.

L'analyse est menée au regard des besoins du processus de prise de décision, et l'obligation de lui fournir des informations représentatives issues de l'analyse de risque, afin d'assurer la robustesse de la décision finale. Les détails techniques des différentes méthodes de présentation d'incertitude sont limités à ce qui est nécessaire pour analyser et juger leurs capacités à communiquer un niveau de risque et les incertitudes associées à des décisionnaires, dans le contexte de l'analyse de risque de systèmes à forts potentiels de dangers en présence d'incertitude.

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Abstract

Title Literature review of methods for representing uncertainty

Keywords uncertainty representation, probability, possibility theory

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This document provides a critical review of different frameworks for uncertainty analysis, in a risk analysis context: classical probabilistic analysis, imprecise probability (interval analysis), probability bound analysis, evidence theory, and possibility theory.

The driver of the critical analysis is the decision-making process and the need to feed it with representative information derived from the risk assessment, to robustly support the decision. Technical details of the different frameworks are exposed only to the extent necessary to analyze and judge how these contribute to the communication of risk and the representation of the associated uncertainties to decision-makers, in the typical settings of high-consequence risk analysis of complex systems with limited knowledge on their behaviour.

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Contents

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Introduction

Context: uncertainty and uncertainty analysis in risk assessment

In all generality, the quantitative analyses of the phenomena occurring in many engineering applications are based on mathematical models which are then turned into operative computer codes for simulation. A model provides a representation of a real system which is dependent on a number of hypotheses and parameters. The model can be deterministic (*e.g.* Newton's dynamic laws or Darcy's law for groundwater flow) or stochastic (*e.g.* the Poisson model for describing the occurrence of earthquake events).

In practice, the system under analysis cannot be characterized exactly: the knowledge of the underlying phenomena is incomplete. This leads to **uncertainty in the analysis**, which can be defined as a *state* of the analyst who cannot describe or foresee a phenomenon due to **intrinsic variability of the phenomenon** itself, or to **lack of knowledge** and information. This leads in practice to uncertainty on both the values of the model parameters and on the hypotheses supporting the model structure. Such uncertainty propagates within the model and causes **variability in its outputs**: for example, when many values are plausible for a model parameter, the model outputs associated to the different values of the uncertain parameter will be different. The quantification and characterization of the resulting output uncertainty is an important issue when models are used to guide decision-making. This topic is known as **uncertainty analysis**.

An uncertainty analysis aims at determining the uncertainty in analysis results that derives from uncertainty in analysis inputs [Helton et al. 2006]. We may illustrate the ideas of the uncertainty analysis by introducing a model $f(\mathbf{Y})$, which depends on the input quantities $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ and on the function f; the quantity of interest Z is computed by using the model $Z = f(\mathbf{Y})$. The uncertainty analysis of Z requires an assessment of the uncertainties about Y and a propagation through the model f to produce an assessment of the uncertainties about Z. Typically, the uncertainty about $\mathbf{Y}$ and the uncertainty related to the model structure f, *i.e.*, uncertainty due to the existence of alternative plausible hypotheses on the phenomena involved, are treated separately; actually, while the first source of uncertainty has been widely investigated and more or less sophisticated methods have been developed to deal with it, research is still ongoing to obtain effective and agreed methods to handle the uncertainty related to the model structure [Parry et Drouin 2009]. See also [Aven 2010c] who distinguishes between **model inaccuracies** (the differences between Z and $f(\mathbf{Y})$), and **model uncertainties** due to alternative plausible hypotheses on the phenomena involved.

Uncertainty is thus an unavoidable aspect of modeling system behaviour, and is particularly significant when attempting to understand their **limits of operation** (reaction to unusually high loads, temperatures, pressures, *etc.*), where less experimental data is typically available. In spite of how much dedicated effort is put into improving the understanding of systems, components and processes through the collection of representative data, the appropriate characterization, representation, propagation and interpretation of uncertainty remains a fundamental element of the risk analysis of any system.

Types of uncertainty

In a risk assessment context, it is convenient to distinguish between "aleatory" and "epistemic" uncertainty [Apostolakis 1990; Helton et Oberkampf 2004; USNRC 2009]. The former refers to phenomena occurring in a random way: probabilistic modeling offers a sound and efficient way to describe such occurrences. The latter captures the analyst's confidence in the model by quantifying the degree of belief of the analysts on how well it represents the actual system; it is also referred to as *state-of-knowledge* or *subjective* uncertainty and can be *reduced* by gathering information and data to improve the knowledge on the system behavior.

Aleatory uncertainties concern, for instance, the occurrence of the events that define the various possible accident scenarios, the time to failure of a component or the random variation of the actual physical dimensions and material properties of a component or system (due to differences between the as-built system and its design upon which the analysis is based) [USNRC 1990; Helton 1998; USNRC 2002]. Two examples of classical probabilistic models used to describe this kind of uncertainties in probabilistic risk assessments (PRA) are the Poisson model for events randomly occurring in time (*e.g.*, random variations of the operating state of a valve) and the binomial model for events occurring "as the immediate consequence of a challenge" (*e.g.*, failures on demand) [USNRC 2005; Hofer et al. 2002; Cacuci et Ionescu-Bujor 2004; Krzykacz-Hausmann 2006].

Epistemic uncertainty is associated with the lack of knowledge about the properties and conditions of the phenomena underlying the behavior of the systems. This uncertainty manifests itself in the model representation of the system behavior, in terms of both (*model*) uncertainty in the hypotheses assumed and (*parameter*) uncertainty in the (fixed but poorly known) values of the parameters of the model [Cacuci et Ionescu-Bujor 2004; Helton et al. 2006]. Both model and parameter uncertainties associated to the current state of knowledge of the system can be represented by subjective probability distributions within a Bayesian approach to PRA [Apostolakis 1990, 1995, 1999].

Whereas epistemic uncertainty can be reduced by acquiring knowledge and information on the system, aleatory uncertainty cannot, and for this reason it is sometimes called *irreducible uncertainty*.

Representing and describing uncertainty

Probabilistic analysis is the most widely used method for characterizing uncertainty in physical systems and models. In the probabilistic approach, uncertainties are characterized by the probabilities associated with events (an event corresponds to any of the possible states a physical system can assume, or any of the possible predictions of a model describing the system).

However, the purely probability-based approaches to risk and uncertainty analysis can be challenged under the common conditions of limited or poor knowledge on the high-consequence risk problem, for which the information available does not provide a strong basis for a specific probability assignment. In such a decision-making context, certain stakeholders may not be satisfied with a probability assessment based on subjective judgments made by a group of analysts. In this view, a broader risk description is sought where all the uncertainties are laid out 'plain and flat', with no additional information inserted in the analytic evaluation in the form of assumptions and hypotheses which cannot be proven right or wrong. This concern has sparked a number of investigations in the field of uncertainty representation and analysis, which has led to the development of alternative frameworks, which can be grouped in four main categories [Aven 2010b, 2011; Aven et Steen 2010; Aven et Zio 2011; Ferson et Ginzburg 1996; Flage et al. 2009]:

1. **imprecise probability**, after [Walley 1991] and the robust statistics area [Berger 1994];
2. **probability bound analysis**, combining probability analysis and interval analysis [Ferson et Ginzburg 1996; Ferson et Hajagos 2004; Ferson et Tucker 2006; Ferson et al. 2007, 2010; Moore 1979];
3. **random sets**, in the two forms proposed by [Dempster 1967] and [Shafer 1976] (but see also [Ferson et al. 2003, 2004; Helton et al. 2007, 2008; Sentz et Ferson 2002]);

4. **possibility theory** [Baudrit et Dubois 2006; Baudrit et al. 2006, 2008; Dubois 2006; Dubois et Prade 1988], which is formally a special case of the imprecise probability and random set theories.

Finally, notice that in the implementation of the decision it is common that the decision-makers seek for further protection by adding conservatisms and performing traditional engineering approaches of 'defense-in-depth' to bound uncertainties and in particular 'unknown unknowns' (completeness uncertainty).

Objectives of this document

In this document, we critically revisit the above mentioned frameworks of uncertainty analysis. The driver of the critical analysis is the decision-making process and the need to feed it with representative information derived from the risk assessment, to robustly support the decision [Aven et Zio 2011]. The technical details of the different frameworks will be exposed only to the extent necessary to analyze and judge how these contribute to the communication of risk and the representation of the associated uncertainties to decision-makers, in the typical settings of high-consequence risk analysis of complex systems with limited knowledge on their behaviour.

Document structure

The remainder of this document is structured as follows:

- Chapter 2 presents probabilistic analysis;
- Chapter 3 presents imprecise probability, after [Walley 1991] and the robust statistics area;
- Chapter 4 discusses probability bound analysis, combining probability analysis and interval analysis;
- Chapter 5 introduces random sets, in the two forms proposed by [Dempster 1967] and [Shafer 1976];
- Chapter 6 presents possibility theory, which is formally a special case of the imprecise probability and random set theories;
- Chapter 7 discusses some practical considerations for decision-making processes, and chapter 8 concludes the analysis.

Readers may be interested in certain other documents by the same authors in the collection of the *Cahiers de la Sécurité Industrielle*:

- *Uncertainty characterization in risk analysis for decision-making practice* (CSI-2012-07), which provides an overview of sources of uncertainty which arise in each step of a probabilistic risk analysis;
- *Overview of risk-informed decision-making processes* (CSI-2012-10), which illustrates the way in which NASA and the US Nuclear Regulatory Commission implement risk-informed decision-making.
- *Case studies in uncertainty propagation and importance measure assessment* (CSI-2013-12), which presents three case studies of component importance measure estimation in the presence of epistemic uncertainties and of the propagation of uncertainties through a risk flooding model.

2

Probability theory

The traditional tool used to express the uncertainties in risk assessments is probability **[Apostolakis 1990]**. In this context, the quantities Z and Y referred to in the introduction could be chances representing fractions in a large (in theory infinite) population of similar items (loosely speaking, a chance is the Bayesian term for a frequentist probability, *cf.* the representation theorem of **[de Finetti 1974]** or **[Bernardo et Smith 2000, p. 172]**). In this case, the assessment is consistent with the so-called probability of frequency approach which is based on the use of subjective probabilities to express epistemic uncertainties of unknown frequencies, *i.e.* the chances **[Kaplan et Garrick 1981]**. The probability of frequency approach constitutes the highest level of uncertainty analysis according to a commonly used uncertainty treatment classification system **[Paté-Cornell 1996]**.

Further details on probability theory are given in the following: in particular, in § 2.1 different interpretations of probability are provided; in § 2.2 one of the available techniques for propagating uncertainty in a probabilistic framework (*i.e.*, Monte Carlo Simulation, or MCS) is described in detail; finally, in § 2.3 the advantages and disadvantages of a probabilistic representation of uncertainty are thoroughly discussed.

2.1 Uncertainty representation

Probability is a single-valued measure of uncertainty, in the sense that uncertainty about the occurrence of an event A is represented by a single number $P(A)$. Different interpretations of probabilities exist, and these are closely related to different notions of uncertainty. Two interpretations of probability are of widespread use in risk analyses: the *relative frequency interpretation* (described in § 2.1.1) and the subjective or *Bayesian interpretation* (§ 2.1.2).

2.1.1 The frequentist view

The relative frequency interpretation of probability

Definition

In this interpretation, probability is defined as the fraction of times an event A occurs if the situation considered were repeated an infinite number of times. Taking a sample of repetitions of the situation, randomness causes the event A to occur a number of times and to not occur the rest of the times. Asymptotically, this process generates a fraction of successes, the "true" probability $P(A)$. This uncertainty (*i.e.*, variation) is sometimes referred to as aleatory uncertainty.

In this context, let Ω be the sample space containing *all* the values that a given random variable Y of interest can assume. In the discrete case, a *discrete* Probability Distribution Function (PDF) $d_Y(y)\colon \Omega \to [0,1]$ exists such that $\sum\limits_{y\in\Omega} d_Y(y) = 1$; in the continuous case, a Probability Density Function (PDF) $p_Y(y)$ exists such that $\int\limits_{\Omega} p_Y(y)\,dy = 1$. The number $d_Y(y)$ represents the (limit) frequency of observing y after many trials in the discrete case, and the density of y in the continuous case. For any measurable subset A of Ω called event, the probability $P(A)$ of A is

$$P(A) = \sum_{y \in A} d_Y(y) \qquad \text{(discrete case)}$$
$$P(A) = \int_A p_Y(y)\mathrm{d}y \qquad \text{(continuous case)}$$

The probability $P(A)$ this defined is required to have the following basic properties **[Helton et Oberkampf 2004]**:

1. if $A \in \Omega$, then $0 \leq P(A) \leq 1$;
2. $P(\Omega) = 1$;
3. if $A_1, A_2, \ldots A_i, \ldots$ is a sequence of *disjoint* sets (events) from Ω, then $P(\cup_i A_i) = \sum_i P(A_i)$;
4. $P(A) = 1 - P(\bar{A})$ (self-duality property): in words, the probability of an event occurring (*i.e.* $P(A)$) and the probability of an event not occurring (*i.e.* $P(\bar{A})$) must sum to one: thus, specification of the likelihood of an event occurring in probability theory also results in, or implies, a specification of the likelihood of that event not occurring[1].

Finally, notice that in the continuous case the Cumulative Distribution Function (CDF) of Y is $F_Y : \Omega \to [0, 1]$, defined from the PDF $p_Y(y)$ as follows:

$$F_Y(y) = P((-\infty, y]) = P(Y \leq y) = \int_{-\infty}^{y} p_Y(t)\, dt, \quad \forall y \in \Omega \tag{2.1}$$

By way of example, let us assume that the random variable Y is normal, *e.g.*, $Y \sim N(5, 0.25)$: the corresponding PDF $p_Y(y)$ and CDF $F_Y(y)$ are shown in figure 2.1, left and right, respectively. The probability that the variable Y is lower than or equal to $y_1 = 5.2$, *i.e.*, $P\{Y \leq y_1 = 5.2\} = \int_{-\infty}^{y_1=5.2} p_Y(y)\, dy = 0.79$ is pictorially shown in figure 2.1 (left) as the shaded area included between the PDF $p_Y(y)$ and the straight line $y_1 = 5.2$; notice that this probability is equal to the value of the CDF $F_Y(y)$ in correspondence of $y_1 = 5.2$, *i.e.*, $F_Y(5.2) = 0.79$ (figure 2.1, right).

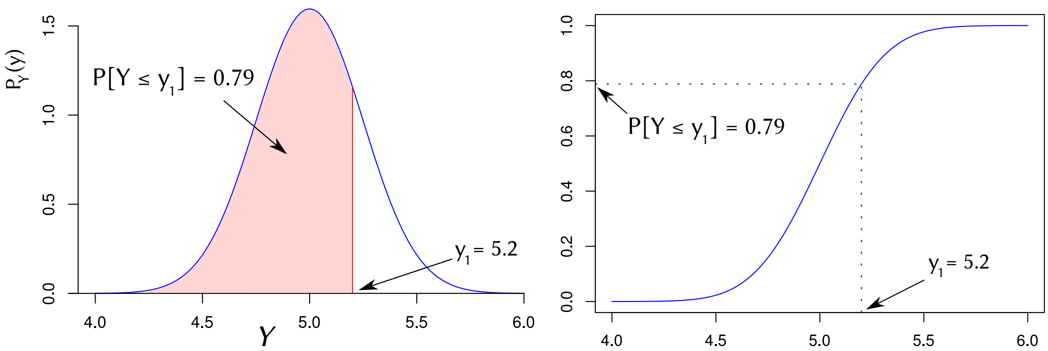

Figure 2.1 – *Probability density function, $p_Y(y)$ (left) and cumulative distribution function $F_Y(y)$ (right) of the normal random variable $Y \sim N(5, 0.25)$*

Referring to the frequentist definition of probability given above, of course in practice it is not possible to repeat the experiment an infinite number of times and thus $P(A)$ needs to be estimated, for example by the relative frequency of occurrence of A in the finite sample considered. The lack of knowledge about the true value of $P(A)$ is termed *epistemic uncertainty*. Whereas epistemic uncertainty can be reduced (by extending the size of the sample), the aleatory uncertainty cannot. For this reason it is sometimes called irreducible uncertainty **[Helton et Burmaster 1996]**.

[1] This property is *peculiar* to probability theory: in general, *less restrictive* conditions on the specification of likelihood are present in evidence and possibility theories (see chapters 5 and 6).

2.1.2 The subjective (Bayesian) view

In the light of this issue, a *subjective (Bayesian) interpretation* of probability can be given where probability is a purely epistemic-based expression of uncertainty as seen by the assigner, based on his/her background knowledge.

The subjective interpretation of probability

DEFINITION

In this view, the probability of an event A represents the *degree of belief* of the assigner with regard to the occurrence of A. The probability can be assigned with reference to either betting or some standard event. If linked to betting, the probability of the event A, $P(A)$, is the price at which the assessor is neutral between buying and selling a ticket that is worth one unit of payment if the event occurs, and is worthless otherwise [de Finetti 1974; Singpurwalla 2006]. Following the reference to a standard, the assessor compares his uncertainty about the occurrence of the event A with some standard events, *e.g.* drawing a favourable ball from an urn that contains $P(A) \times 100\%$ favourable balls [Lindley 2000].

Irrespective of reference, all subjective probabilities are seen as conditioned on the background knowledge $\mathcal{K}$ that the assignment is based on. They are probabilities in the light of current knowledge [Lindley 2006]. To show the dependencies on $\mathcal{K}$ it is common to write $P(A|\mathcal{K})$, but often $\mathcal{K}$ is omitted as the background knowledge is tacitly understood to be a basis for the assignments. Elements of $\mathcal{K}$ may be uncertain and seen as unknown quantities, as pointed out by [Mosleh et Bier 1996]. However, the entire $\mathcal{K}$ cannot generally be treated as an unknown quantity and removed using the law of total probability, *i.e.* by taking $E_{\mathcal{K}}[P(A|\mathcal{K})]$ to obtain an unconditional $P(A)$.

In this view, randomness is not seen as a type of uncertainty in itself. It is seen as a basis for expressing epistemic-based uncertainty. A relative frequency generated by random variation is referred to as a chance, to distinguish it from a probability, which is reserved for expressions of epistemic uncertainty based on belief [Singpurwalla 2006; Lindley 2006]. Thus, we may use probability to describe uncertainty about the unknown value of a chance. As an example, consider an experiment in which the event A of interest occurs $p \times 100\%$ of the times the experiment is performed. Suppose that the chance p is unknown. Then, the outcomes of the experiment are not seen as independent, since additional observations would provide more information about the value of p. On the contrary, in the case that p were known the outcomes would be judged as independent, since nothing more could be learned about p from additional observations of the experiment. Thus, conditional on p the outcomes are independent, but unconditionally they are not; they are exchangeable. The probability of an event A for which p is known is simply p. In practice, p is in most cases not known, and the assessor expresses his/her (*a priori*) uncertainty about the value of p by a probability distribution $H(p)$. Then, the probability of A can be expressed as

$$P(A) = \int P(A|p)\,dH(p) = \int p\,dH(p) \tag{2.2}$$

One common approach to risk analysis is to use epistemic-based probabilities to describe uncertainty about the true value of a relative frequency-interpreted probability (chance). This is called the **probability of frequency approach** [Kaplan et Garrick 1981] – probability referring to the epistemic-based expressions of uncertainty and frequency to the limiting relative frequencies of events. By taking the expected value of the relative frequency-based probability with respect to the epistemic-based probabilities, both aleatory and epistemic uncertainties are reflected.

2.2 Uncertainty propagation

Referring to the uncertainty propagation task framed in the previous section, let us consider a model whose output is a function $Z = f(\mathbf{Y}) = f(Y_1, Y_2, \ldots, Y_j, \ldots, Y_n)$ of n uncertain variables $Y_j, j \in 1, 2, \ldots, n$, that are "probabilistic", *i.e.*, their uncertainty is described by probability distributions $p_{Y_1}(y_1), p_{Y_2}(y_2), \ldots, p_{Y_j}(y_j), \ldots, p_{Y_n}(y_n)$. In such a case, the propagation of uncertainty can be performed by Monte Carlo Simulation (MCS), which comprises the following two main steps **[Kalos et Whitlock 1986; Marseguerra et Zio 2002]**:

1. repeated random sampling of possible values of the uncertain variables $Y_j, j = 1, 2, \ldots, n$;
2. evaluation of the model function $Z = f(\mathbf{Y}) = f(Y_1, Y_2, \ldots, Y_j, \ldots, Y_n)$ in correspondence of all the values sampled at step i. above.

In more detail, the operative steps of the procedure are:

1. set $i = 1$;
2. sample the i$^{\text{th}}$ set of random realizations y_j^i, $j = 1 \ldots n$, of the "probabilistic" variables Y_j, $j = 1, 2, \ldots, n$, from the corresponding probability distributions $p_Y^1(y^1), p_Y^2(y^2), \ldots, p_Y^j(y_j j), \ldots, p_Y^n(y_n)$;
3. calculate the value z^i of the model output Z as $z^i = f(y_1^i, y_2^i, \ldots, y_j^i, \ldots, y_n^i)$;
4. if $i < N$ (N being the number of samples[2]), set $i = i + 1$ and go back to step 2. above; otherwise, go to 5. below;
5. post-process the N output values $z_i = f(y_1^i, y_2^i, \ldots, y_j^i, \ldots, y_n^i)$, $i \in 1 \ldots N$, thereby obtained in order to provide an *empirical* estimate for
   - ▷ the Probability Density Function (PDF) $p_Z(z)$ of Z (*e.g.*, by tallying the resulting values $z_i = f(y_1^i, y_2^i, \ldots, y_j^i, \ldots, y_n^i)$, $i = 1 \ldots N$, in a *histogram*);
   - ▷ the Cumulative Distribution Function (CDF) $F_Z(z)$ of Z as $F_Z(z) \approx \frac{1}{N} \sum_{i=1}^{N} I\{z^i \leq z\}$, where $I\{z^i \leq z\}$ is 1, if $z^i < z$ and 0, otherwise.

Finally, notice that the random sampling performed at step 2. above may account for possible dependencies existing between the uncertain variables $Y_j, j = 1 \ldots n$; on the other hand, such dependencies can be obviously included in the analysis, *only* if they can be modeled within a classical MCS framework **[Ferson 1996a,b]**.

By way of example and only for illustration purposes, let Y_1 be represented by a uniform probability distribution $U[a_1, b_1]$, where $a_1 = 1$ and $b_1 = 3$ (figure 2.2, top, left), and Y_2 be represented by a uniform probability distribution $U[a_2, b_2]$, where $a_2 = 2$ and $b_2 = 5$ (figure 2.2, top, right). Figure 2.2, bottom, left shows the *analytical* PDF $p_Z(z)$ of the output $Z = Y_1 + Y_2$ (solid line) together with the corresponding *empirical* estimate (histogram) obtained by MCS with $N = 100\,000$ samples; figure 2.2, bottom, right shows the empirical estimate of the CDF $F_Z(z)$ of $Z = Y_1 + Y_2$ (solid line) obtained by MCS with $N = 100\,000$ samples.

2.3 Discussion

The probability-based approaches to risk and uncertainty analysis can be challenged. Many researchers and analysts find the above framework for assessing risk and uncertainties to be too narrow: risk is more than some analysts' subjective probabilities, which may lead to poor predictions. The knowledge that the probabilities are based on could be poor and/or based on wrong assumptions. Many analysts would argue that the information available for the probabilities commonly does not provide a sufficiently strong basis for a specific probability assignment. In a risk analysis context, there are often many stakeholders and they may not be satisfied with a probability-based assessment providing subjective judgments made by one analysis group. A broader risk description is sought.

[2] The number of samples used in a MCS is chosen as a compromise between the cost of the simulation (a larger number of samples being more expensive to evaluate) and the level of confidence obtained (a larger statistical sample leads to a higher level of confidence). The number of samples can be chosen before running the simulation ($N = 10\,000$, for example) or can be determined during the simulation, by continuing to sample random realizations until the quantity of interest is no longer changing significantly.

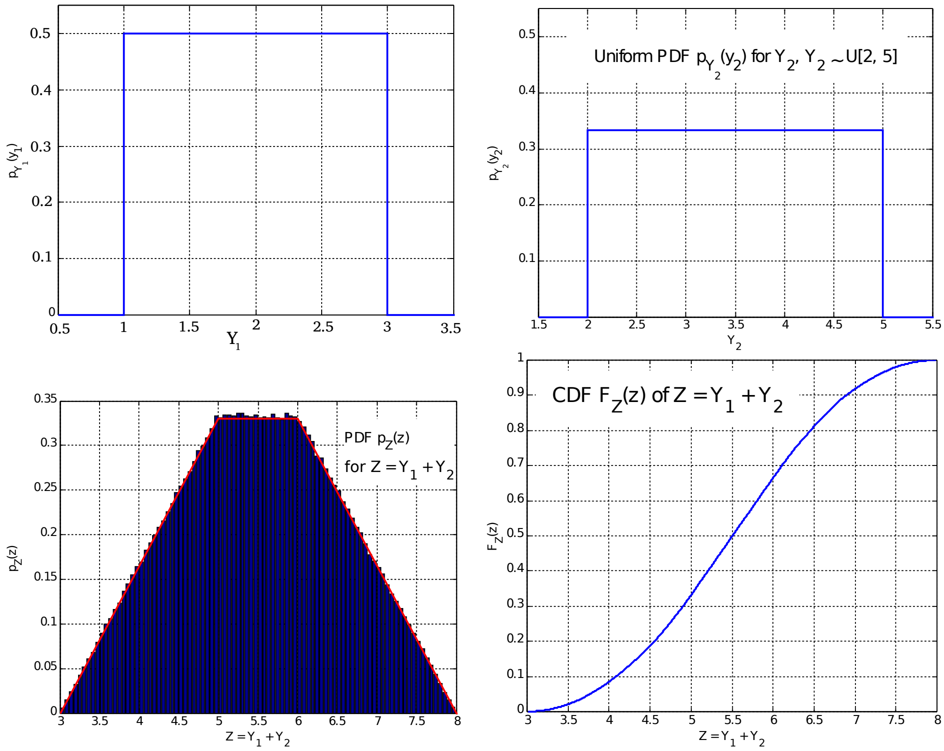

Figure 2.2 – *Top, left: uniform PDF* $U[a_1, b_1]$, *where* $a_1 = 1$ *and* $b_1 = 3$, *for uncertain variable* Y_1; *top, right: uniform PDF* $U[a_2, b_2]$, *where* $a_2 = 2$ *and* $b_2 = 5$, *for uncertain variable* Y_2; *bottom, left: analytical PDF* $p_Z(z)$ *of the output* $Z = Y_1 + Y_2$ *(solid line) together with the corresponding empirical estimate (histogram) obtained by MCS with* $N = 100\,000$ *samples; bottom, right: empirical estimate of the CDF* $F_Z(z)$ *of the output* $Z = Y_1 + Y_2$ *(solid line) obtained by MCS with* $N = 100\,000$ *samples*

Adopting the subjective probability approach, probabilities can always be assigned, but their support is not reflected by the numbers produced. [Dubois 2010] expresses the problem in this way: if the ill-known inputs or parameters to a mathematical model are all represented by single probability distributions, either objective when available or subjective when scarce information is available, then the resulting distribution on the output can hardly be properly interpreted: "the part of the resulting variance due to epistemic uncertainty (that could be reduced) is unclear".

The problem seems to be that aleatory uncertainties are mixed with epistemic uncertainties. However, if chances (more generally, probability models with parameters) can be established (justified) reflecting the aleatory uncertainties a full risk description needs to assess uncertainties about these quantities. It would not be sufficient to provide predictive distributions alone, as important aspects of the risk then would not be revealed. The predictive distributions would not distinguish between the stochastic variation and the epistemic uncertainties as noted by [Dubois 2010]. The indicated inadequacy of the subjective probabilities for reflecting uncertainties is thus more an issue of addressing the right quantities: if chances can be established (justified), the subjective probabilities should be used to reflect the uncertainties about these chances.

Probability models constitute the basis for statistical analysis, and are considered essential for assessing the uncertainties and drawing useful insights [Helton 1994; Winkler 1996]. The probability models coherently and mechanically facilitate the updating of probabilities. A probability model presumes some sort of model stability, populations of similar units need to be constructed (in the Bayesian context, formally an infinite set of exchangeable random variables). But such stability is often not fulfilled [Bergman 2008]. Consider the definition of a chance. In the case of a die we would establish a probability model expressing that the distribution of outcomes is given by $(p_1, p_2, \ldots, p_6)$, where p_i is the chance of outcome i, interpreted as the fraction of outcomes resulting in outcome i. However, in a risk assessment context the situations are often unique, and the establishment of chances means the construction of fictional populations of non-existing similar situations. Then chances and probability models in general, cannot be easily defined as in the die tossing example; in many cases, they cannot be meaningfully defined at all. For example, it makes no sense to define a chance (frequentist probability) of a terrorist attack [Aven et Heide 2009]. In other cases, the conclusion may not be so obvious. For example, a chance of an explosion scenario in a process plant may be introduced in a risk assessment, although the underlying population of infinite similar situations is somewhat difficult to describe.

There is a huge literature addressing the problems of the probability–based risk assessments. Here are some examples of critical issues raised. [Reid 1992] argues that there is a common tendency of underestimation of the uncertainties in risk assessments. The disguised subjectivity of risk assessments is potentially dangerous and open to abuse if it is not recognized. According to [Stirling 2007], using risk assessment when strong knowledge about the probabilities and outcomes does not exist, is irrational, unscientific and potentially misleading. [Tickner et Kriebel 2006] stress the tendency of decision-makers and agencies not to talk about uncertainties underlying the risk numbers. Acknowledging uncertainty can weaken the authority of the decision-maker and agency, by creating an image of being unknowledgeable. Precise numbers are used as a facade to cover up what are often political decisions. [Renn 1998] summarizes the critique drawn from the social sciences over many years and concludes that technical risk analyses represent a narrow framework that should not be the single criterion for risk identification, evaluation and management.

Summing up, the **advantages of Monte Carlo Simulation** (MCS) (as a means of propagating uncertainty in classical probability theory) are the following [Ferson 1996a,b, 1999; Ferson et al. 2010]):

- it is *flexible* because it does not suffer from the *complexity*, *multidimensionality* and *nonlinearity* of the system model $f(\mathbf{Y})$ and, therefore, it does not force to resort to simplifying approximations;
- it is *simple* to implement and explain;
- it can use information about *correlations* among variables.

On the contrary, the **disadvantages of MCS** can be summarized as follows [Ferson 1996a,b; Ferson et Burgman 1995; Ferson et Ginzburg 1996; Ferson et Long 1994; Ferson 1999; Ferson et al. 2010]:

- although it is not widely acknowledged, it requires a lot of *empirical information.* Probability distribution of each of the variables involved in the assessment need to be estimated. This means knowing not only their means, but also their variances and indeed the shapes of their statistical distributions: in other words, the analyst is asserting that he/she can estimate the probabilities of all possible values for every uncertain parameter. Also, the analyst needs to estimate the statistical dependencies among all of these variables: this means knowing the cross correlations among the variables and, in principle, any triplewise or higher-order interactions that may exist. Formally speaking, MCS cannot be done without all this information [Ferson 1996b];
- when all the required empirical information is not available, the analyst is forced to make (*arbitrary*, *subjective* and often *unjustified*) *assumptions* and *guesses*;
- the assumptions that are routinely made about distribution shapes and statistical dependencies between distributions can lead to *'non-protective'* conclusions that *underestimate* the true *risks.* For instance, analysts often assume all their input variables are mutually independent and pretend that all the correlations would be zero if measured: the result of such an assumption (if it is wrong) can be to miscalculate risk [Ferson 1994a; Ferson et Burgman 1995; Ferson et Ginzburg 1996; Ferson et Long 1994]. In fact, the tails of the resulting probability distributions can be very sensitive to information about the shapes and dependencies of statistical distributions [Bukowski et al. 1995; Ferson 1994a]. It is the tails that give the probabilities of extreme events. An argument can be made that the whole point of the risk analysis in the first place is to learn about these extreme events in the tails. Yet the detailed data needed to get reasonable estimates of them in a probabilistic risk analysis are hardly ever available in practical situations. When the assumptions made in

a Monte Carlo Simulation are wrong, it is possible to estimate risks to be low when they are actually high. In this sense, probabilistic risk analysis as it is currently practiced may have the opposite flaw compared to the hyper-conservatism of worst case approaches;

- ▷ it confounds ignorance with variability because it has an *inadequate model* of *ignorance*. For instance, when all that is known about a quantity is its theoretical range, probabilists traditionally employ a uniform distribution over this range to represent this uncertainty. This approach dates back to Laplace himself and his “principle of insufficient reason”. The approach is also justified by modern reasoning appealing to the “maximum entropy criterion” [Jaynes 1957; Lee et Wright 1994]. *But not knowing the value of a quantity is not the same as having it vary randomly.* When probabilists do not distinguish between equiprobability and ignorance, they are confounding variability with incertitude.
- ▷ there is *no* sound and satisfactory way to handle uncertainty about the proper mathematical model to use (*i.e.*, to answer questions such as “is this the correct expression to evaluate in the first place? Are these assumptions appropriate?”);
- ▷ *merging* subjective estimates coming from *different sources* may not provide reliable results;
- ▷ Monte Carlo Simulation may be *computationally cumbersome*. Actually, it is based on the repeated random sampling of possible values of the parameters **Y** and the subsequent evaluation of the model output $Z = f(\mathbf{Y})$ in correspondence of *each* of these sampled values. However, two factors increase the computational effort required, sometimes making it impractical [Schueller 2009]:
  - a **large number of samples** (*e.g.*, of the order of many thousands) is usually necessary to achieve acceptable estimation accuracy (in particular, when the quantities to be estimated are very small probabilities);
  - **long calculations** (several hours) are typically necessary for each run of the system model $f(\mathbf{Y})$ (this is particularly true if the model function $f(\mathbf{Y})$ is represented by a detailed mechanistic code).

Given the above critiques, it is not surprising that alternative approaches for representing and describing uncertainties in risk assessment have been suggested, such as the four categories described in the following chapters.

3

Imprecise (interval) probability

In § 3.1, the concept of imprecise (or interval) probabilities, also known as interval analysis in the literature, is explained. In § 3.2, interval arithmetic is presented as a means of propagating uncertainty within the framework of interval analysis; finally, in § 3.3, the advantages and disadvantages of the interval analysis framework for uncertainty representation and propagation are thoroughly discussed.

3.1 Uncertainty representation

To explain the meaning of imprecise probabilities (or interval probabilities) consider an event A. Uncertainty on the likelihood of A occurring is represented by a lower probability $P(A)$ and an upper probability $\overline{P}(A)$, giving rise to a probability interval $[P(A), \overline{P}(A)]$, where $0 \leq P(A) \leq \overline{P}(A) \leq 1$. The difference

$$\Delta P(A) = \bar{P}(A) - \underline{P}(A) \tag{3.1}$$

is called the *imprecision* in the representation of the event A. Single-valued probabilities are a special case of no imprecision and the lower and upper probabilities coincide.

Peter M. Williams developed a mathematical framework for imprecise probabilities, based on de Finetti's betting interpretation of probability **[de Finetti 1974]**. This foundation was further developed independently by Vladimir P. Kuznetsov and Peter Walley (the former only published in Russian), see **[Kuznetsov 1991]** and **[Walley 1991]**. Following de Finetti's betting interpretation, the lower probability is interpreted as the maximum price for which one would be willing to buy a bet which pays 1 if A occurs and 0 if not, and the upper probability as the minimum price for which one would be willing to sell the same bet. If the upper and lower values are equal, the interval is reduced to a precise probability. These references, and **[Walley 1991]** in particular, provide an in-depth analysis of imprecise probabilities and their interpretations, with a link to applications to probabilistic reasoning, statistical inference and decisions.

It is however also possible to interpret the lower and upper probabilities using the reference to a standard interpretation of a subjective probability $P(A)$: such an interpretation is indicated by **[Lindley 2006, p. 36]**. Consider the subjective probability $P(A)$ and say that the analyst states that his/her assigned degree of belief is greater than the urn chance of 0.10 (the degree of belief of drawing one particular ball from an urn which include 10 balls) and less than the urn chance of 0.5. The analyst is not willing to make any further judgment. Then, the interval $[0.10, 0.50]$ can be considered an imprecision interval for the probability $P(A)$.

Of course, even if the assessor assigns a probability $P(A) = 0.3$, one may interpret this probability as having an imprecision interval [0.26, 0.34] (as a number in this interval is equal to 0.3 when displaying one digit only), interpreted analogously to the [0.1, 0.5] interval. Hence imprecision is always an issue in a practical uncertainty analysis context. This imprecision is commonly viewed as a result of measurement problems. **[Lindley 2006]** argues that the use of interval probabilities confuses the *concept* of measurement with the *practice* of measurement. The reference to the urn lottery provides a norm, and measurement problems may make the assessor unable to behave according to it. See also discussion in **[Bernardo et Smith 2000, p. 32]**.

However, other researchers and analysts have a more positive view on the need for such intervals, see discussions in [Aven et Zio 2011; Ferson et Ginzburg 1996; Ferson et Hajagos 2004; Ferson et Tucker 2006; Ferson et al. 2007, 2010]: imprecision intervals are required to reflect phenomena as discussed above, for example when experts are not willing to express their knowledge more precisely than by using probability intervals.

Imprecise probabilities are also linked to the relative frequency interpretation of probability [Coolen et Utkin 2007]. The simplest case reflects that the "true" frequentist probability p is in the interval $[\underline{P}(A), \overline{P}(A)]$ with certainty. More generally and in line with the above interpretations of imprecision intervals based on subjective probabilities $P(\cdot)$, a two-level uncertainty characterization can be formulated (see, *e.g.*, [Kozine et Utkin 2002]): $[\underline{P}(A), \overline{P}(A)]$ is an imprecision interval for the subjective probability $P(a \leq p \leq b)$ where a and b are constants. In the special case that $\underline{P}(A) = \overline{P}(A)$ (= q, say) we are led to the special case of a $q \times 100\%$ *credibility interval* for p (*i.e.*, with subjective probability q, the true value of p is in the interval $[a, b]$).

3.2 Uncertainty propagation

[Moore 1966] described the use of *interval arithmetic* to evaluate the ranges of functions taking interval arguments. The approach consists in generalizing the definitions of the binary operations to the case in which the inputs are intervals. In practice, the rules for interval arithmetic are obtained from answering the following question: "what are the largest and smallest possible values that could be obtained under this mathematical operation?" [Ferson et al. 2004]. For all real numbers x, y, z, t, such that $0 \leq x \leq y \leq 1$ and $0 \leq z \leq t \leq 1$, we have:

$$[x, y] + [z, t] = [x + z, y + t] \tag{3.2}$$
$$[x, y] - [z, t] = [x - t, y - z] \tag{3.3}$$
$$[x, y] \times [z, t] = [x \times z, y \times t] \tag{3.4}$$
$$[x, y]/[z, t] = [x/t, y/z], \mathit{with}\ z > 0 \tag{3.5}$$
$$\min([x, y], [z, t]) = \big[min(x, z), min(y, t)\big] \tag{3.6}$$
$$\max([x, y], [z, t]) = \big[max(x, z), max(y, t)\big] \tag{3.7}$$

Note that formulas (3.4) and (3.5) for multiplication and division, respectively, are considerably simpler than those of ordinary interval arithmetic. The simplicity is a consequence of the constraint that *probabilities* must lie in the interval $[0, 1]$ (*i.e.*, $0 \leq x \leq y \leq 1$ and $0 \leq z \leq t \leq 1$). In the case of real numbers ranging in $(-\infty, +\infty)$, the general formulas for multiplication and division become more complicated so that

$$[x, y] \cdot [z, t] = \big[min(x \cdot z, x \cdot t, y \cdot z, y \cdot t), max(x \cdot z, x \cdot t, y \cdot z, y \cdot t)\big] \tag{3.8}$$
$$[x, y]/[z, t] = \big[min(x/z, x/t, y/z, y/t), max(x/z, x/t, y/z, y/t)\big], \quad \mathit{with}\ z, t \neq 0 \tag{3.9}$$

By way of example, let us suppose that the epistemic uncertainty associated to the probabilities $P(A)$ and $P(B)$ of the independent events A and B is represented by the intervals $[0.1, 0.2]$ and $[0.15, 0.35]$, respectively. Using (3.3) and (3.4) the probability $P(A \cup B) = 1-(1-P(A))(1-P(B))$ of the event $(A \cup B)$ is computed as $[1-(1-0.1)(1-0.15), 1-(1-0.2)(1-0.35)] = [0.2350, 0.4800]$.

In many problems, interval arithmetic can be used in a straightforward way to obtain results that are both rigorous and best possible. However, when an uncertain number appears *more than once* in a mathematical expression, the naïve sequential application of the rules of interval arithmetic may yield results that are *wider* than they should be. The result is still *rigorous* in the sense that it is sure to enclose the true range, but it may *fail* to be *best-possible* if it is wider than needed. The reason for this loss of optimality is basically that the uncertainty in the *repeated* parameter is entered into the calculation *more than once* [Ferson 1996a; Ferson et al. 2004]. Referring to the example above, the computation of $P(A \cup B)$ as $P(A) + P(B) - P(A)P(B)$

actually leads to $[0.1 + 0.15 - 0.2 \times 0.35, 0.2 + 0.35 - 0.1 \times 0.15] = [0.1800, 0.5350]$. As expected, the interval $[0.1800, 0.5350]$ obtained using the mathematical expression $[P(A) + P(B) - P(A)P(B)]$ (where both $P(A)$ and $P(B)$ appear *more than once*) is (unnecessarily) much wider than the interval $[0.2350, 0.4800]$ obtained using the expression $[1 - (1 - P(A))(1 - P(B))]$ (where $P(A)$ and $P(B)$ appear *just once*).

Until now it has been assumed that the model function to be evaluated is made of *simple arithmetic operations*, *e.g.*, sums, products, differences, *etc.*, and is *known explicitly*. This is *not* always the case in risk assessment. First, there may be functions that cannot be conveniently characterized as the composition of simple arithmetic functions; second, the form of the function may not even be mathematically known. For example, in many situations, the analyst may need to use a *computer model* whose internal details are not accessible or too complex to analyze: in these cases, the function is a 'black box' to which one gives inputs and from which one collects the outputs **[Ferson 1999]**.

When these 'black-box' functions have to be evaluated by interval analysis, one must be careful not to assume that the upper bound of the function is given by the upper bound of its input(s). By way of example, in expressions like $Z = \frac{2}{Y}$, the *lower* (*upper*) bound of Y should be used to estimate an *upper* (*lower*) bound for Z **[Ferson 1999]**.

In addition, it must be remembered that the function may not even be *monotonic*. If the ranges of the inputs happen to straddle a function minimum, then either the upper or the lower bound of the input will produce the upper bound once it has been mapped through the function, but it cannot be predicted which without knowing all the details. If the input ranges straddle a function maximum, then neither the upper nor the lower bound of the input will yield the upper bound of the function result: some value between them produces the upper bound. In these cases, the endpoints of the input interval cannot be simply used to find the endpoints of the function. By way of example, consider the function $Z(Y) = (Y - 1)^2$, with $Y = [0, 3]$. The lower and upper bounds on Z are obviously 0 (obtained in correspondence of $Y = 1$) and 4 (obtained in correspondence of the upper bound on Y, *i.e.*, $Y = 3$, respectively; however, the naïve identification of the lower and upper bounds on Z through the evaluation of Z in correspondence of the lower and upper bounds of Y would *erroneously* lead to $Z(Y = 0) = 1$ and $Z(Y = 3) = 4$, respectively. Non-monotonicity requires the analysis of the entire input interval. A brute-force approach analogous to a Monte Carlo Simulation (MCS) would be useful. In this approach, the inputs have to be varied over their possible ranges and in all possible combinations, just as in MCS; then, the smallest and largest answers obtained have to be recorded: such values define the interval of the result. This brute-force approach is *approximate* since the true bounds on the result are *not* absolutely *guaranteed* (*e.g.*, due to the presence of cusps or other elements that escape detection in the numerical application) **[Ferson 1999]**.

3.3 Discussion

Based on the considerations and examples above, the **advantages of interval analysis** can be summarized as follows **[Ferson 1999; Ferson et al. 2010]**):

- ▷ it is quite *natural* for scientists who are accustomed to reporting their measured values in terms of the best estimate and the possible error of the estimate. This may be expressed with the 'plus or minus' convention, or in terms of an interval which is presumed to contain the actual value;
- ▷ it is very *simple* and *easy* to explain;
- ▷ it works no matter what the *nature* and *sources* of uncertainty are. In risk assessment practices, measurements are often very expensive to make: as a consequence, there are typically very few of them. It is not uncommon, for instance, that there is but one measurement taken in the field. In this case, since we have no way of even roughly estimating statistical variance, interval arithmetic may be an appropriate way to propagate the estimate's measurement error. In such cases, the size of measurement error can be inferred from the measurement protocol directly. For instance, in using a ruler to measure a length, the measurement error is plus or minus one half of the smallest graduation on the ruler, or perhaps one tenth if you trust your interpolation skills. Such measurement error can be thought of *non-statistical* uncertainty since no probability statement is

involved in its estimation or interpretation. Other forms of non-statistical uncertainty arise when a modeler uses intuition to assign parameter values. They are often given as ranges of feasibility without any further specification. In such cases, no sampling has occurred and no probability is defined (or is even contemplated), so it seems inappropriate to use probabilistic methods and so confound the modeler's ignorance with real statistical variation that may be represented elsewhere in the model;

- ▷ it is especially useful in preliminary *screening* assessments;
- ▷ it is a fairly *straightforward* and *standard* method that yields *rigorous* results. In other words, if the inputs *really* lie within their respective intervals and the mathematical expression used to combine them together is the *real* one, using interval analysis guarantees us that the *true* result will lie somewhere in the computed interval;
- ▷ it obtains rigorous results without the need of any assumptions about the *distributions* or *dependence* among the variables: sure bounds on risk are computed no matter what is true about correlations or the details of the distribution of each variable within its range[3].

On the contrary, the **disadvantages of interval analysis** can be summarized as follows [Ferson 1999; Ferson et al. 2010]:

- ▷ the computed ranges can grow *very quickly*. It can be demonstrated that intervals become more and more conservative as arithmetic operations are applied: as a consequence, intervals become wider and wider and are less and less precise about the result;
- ▷ it *cannot* take account of *distributions*, *correlations/dependencies* and detailed *empirical information* (which may be sometimes available) about a quantity beside its potential range. It would not help, for instance, to know that most values are close to some central tendency, or that the variation in the quantity expressed through time follows a normal distribution. Knowledge about the statistical associations between variables is also useless in interval analyses. Because this method does not use all the available information, it can produce results that are *more conservative* than is necessary given what is known;
- ▷ it addresses only the *bounds* on risks, but it makes no statement about *how likely* such extreme risks are. Even if the upper bound represents an intolerable risk, if the chance of it actually occurring is vanishingly small, it may be unreasonable to base regulation on this value;
- ▷ it is somewhat *paradoxical* in nature because it does *not* compute the *exact value* of a given quantity, but it produces *exact bounds* on that quantity.
- ▷ *repeated* parameters may represent a problem.

The considerations above highlight the need to choose between robust methods that are not very informative and methods that can provide many details but which need an impractical amount of information. Ideally, a method for uncertainty representation and propagation should bring together the robustness and rigor of interval analysis with the detail of probabilistic analysis without making too many empirical demands. Probability bound analysis, described in the next chapter, represents a step forward in this direction.

[3] Notice that this concept can be viewed as an advantage and a disadvantage of the approach at the same time (see below).

4

Probability bound analysis

In probability bound analysis, interval analysis is used for those components whose aleatory uncertainties cannot be accurately estimated; for the other components, traditional probabilistic analysis is carried out. Further details on probability bound analysis are given in what follows: in particular, in § 4.1 the main characteristics of this approach are briefly summarized; in § 4.2 the associated technique for uncertainty propagation is described in detail; finally, in § 4.3 the advantages and disadvantages of probability bound analysis are thoroughly discussed.

4.1 Uncertainty representation

[Ferson et Ginzburg 1996] suggest a combined probability analysis-interval analysis, referred to as a probability bound analysis. The setting is a risk assessment where the aim is to express uncertainties about some parameters $Y_j, j = 1...n$, of a model (a function Z of the Y_j's, for example Z equal to the product of the parameters Y_j). For the parameters where the aleatory uncertainties can be accurately estimated, traditional probability theory is employed; for those parameters where the aleatory uncertainties cannot be accurately determined, interval analysis is used. In this way uncertainty propagation is carried out in the traditional probabilistic way for some parameters, and intervals are used for others. More specifically it means that:

1. for parameters Y_j where the aleatory uncertainties cannot be accurately estimated, use interval analysis expressing that $a_j \leq Y_j \leq b_j$ for constants a_j and b_j;
2. for parameters Y_j where the aleatory uncertainties can be accurately assessed, use probabilities (relative frequency-interpreted probabilities) to describe the distribution over Y_j. In particular, notice that probability bound analysis employs Cumulative Distribution Functions (CDFs) rather than Probability Density Functions (PDFs) to describe aleatory uncertainty and perform computations **[Ferson et Ginzburg 1996; Ferson et al. 2010]**.

4.2 Uncertainty propagation

Uncertainty propagation is performed by combining steps 1. and 2. of § 4.1 above to generate a probability distribution over Z, for the different interval limits. For example, assume that for $j = 1$, interval analysis is used with bounds $a_2 = 2$ and $b_1 = 7$ (figure 4.1, top, left): this interval represents a quantity about whose value the analyst is uncertain because he/she has no more precise measurement of it. It may be varying within this range, or it may be a fixed, unvarying value somewhere within the range: the analyst does not have any particular information one way or the other. On the contrary, for $j = 2$, a probabilistic analysis is used: in particular, Y_2 is described by a lognormal probability distribution $p_{Y_2}(y_2) = ln(\mu_2, \sigma_2)$, where $\mu_2 = 1.6094$ and $\sigma_2 = 0.4214$ (figure 4.1, top, right depicts the corresponding CDF $F_{Y_2}(y_2)$ truncated at the 0.005^{th} and 0.995^{th} percentiles for convenience).

The remaining graphs in figure 4.1 show the product $Z = Y_1 \times Y_2$ (figure 4.1, middle, left), sum $Z = Y_1 + Y_2$ (figure 4.1, middle, right), quotient $Z = Y_2/Y_1$ (figure 4.1, bottom, left) and difference $Z = Y_2 - Y_1$ (figure 4.1, bottom, right). For illustration purposes, let us consider the product $Z = Y_1 \times Y_2$ depicted in figure 4.1, middle, left. The answer that probability bounds analysis produces is *not* a *single* CDF $F_Z(z)$, but rather it is a *region* within which the cumulative probability distribution of the product $Z = Y_1 \times Y_2$ must lie: this region is identified by the upper and lower CDFs $\bar{F}_Z(z)$ and $\underline{F}_Z(z)$, respectively, depicted in figure 4.1, middle, left and it is referred to in the literature as *probability box* (p-box). This is to say that, whatever the true value(s) of the uncertain quantity Y_1 the analyst has represented with the interval, the CDF $F_Z(z)$ of the product $Z = Y_1 \times Y_2$ lies somewhere within the region $[\bar{F}_Z(z), \underline{F}_Z(z)]$ identified by the upper and lower CDFs $\bar{F}_Z(z)$ and $\underline{F}_Z(z)$, respectively, depicted in figure 4.1, middle, left. From a merely computational viewpoint, it is worth noting that the limiting upper CDF $\bar{F}_Z(z)$ on $Z = Y_1 \times Y_2$ is obtained by setting Y_1 to its lower bound $a_1 = 2$ and letting Y_2 vary according to its (aleatory) probability distribution, *i.e.*, $p_{Y_2}(y_2) = ln(\mu_2 = 1.6094, \sigma_2 = 0.4214)$; on the contrary, the limiting lower CDF $\underline{F}_Z(z)$ on $Z = Y_1 \times Y_2$ is obtained by setting Y_1 to its upper bound $b_1 = 7$ and letting Y_2 vary according to its (aleatory) probability distribution, *i.e.*, $p_{Y_2}(y_2) \sim ln(\mu_2 = 1.6094, \sigma_2 = 0.4214)$[4]. The result obtained and in displayed in figure 4.1, middle, left fully expresses the uncertainty induced by the two factors Y_1 (*i.e.*, purely epistemic uncertainty) and Y_2 (*i.e.*, purely aleatory uncertainty). Any more precise answer than the one represented in figure 4.1, middle, left would simply be underestimating the degree of uncertainty present in the calculation of $Z = Y_1 \times Y_2$. For instance, if the analyst had used a uniform distribution $p_{Y_1}(y_1) = U[a_1 = 2, b_1 = 7]$ to represent the first factor Y_1 rather than an interval, and performed the multiplication according to the rules of probability theory, he/she would have obtained *one* particular CDF $F_Z(z)$ *roughly centered* between the upper and lower CDFs $\bar{F}_Z(z)$ and $\underline{F}_Z(z)$ depicted in figure 4.1, middle, left. But such an answer would, however, have a wholly *unjustified precision*. In other words, it might be wrong, either under- or over-estimating probabilities for the possible range of products. Of course it might be exactly correct by accident, but such an outcome would actually be remarkably unlikely.

As highlighted above, a p-box $[\bar{F}_Z(z), \underline{F}_Z(z)]$ is designed to simultaneously express both *variability* and *incertitude*. The horizontal span of the probability bounds are a function of the *variability* in the result (*i.e.*, of aleatory uncertainty); the vertical breadth of the bounds is a function of the analyst's *ignorance/incertitude* (*i.e.*, of epistemic uncertainty). A pure risk analysis problem with perfectly characterized probability distributions as inputs will yield a pure probability distribution as the result. Values, distributions and dependencies that are imperfectly known contribute to a widening of the bounds. The greater the ignorance, the wider the vertical distance between bounds, and the more difficult to make precise probabilistic statements about the expected frequencies of extreme events. But this is what one wants; after all, ignorance should muddle the answer to some extent. Something is obviously amiss information-theoretically if we can combine ignorance and gain more precision than we started with.

Finally, notice that probability distributions, intervals and scalar numbers are all special cases of p-boxes. Because a probability distribution expresses variability and lacks incertitude, the upper and lower bounds of its p-box, $\bar{F}_Z(z)$ and $\underline{F}_Z(z)$, are coincident at the value of the cumulative distribution function (which is a non-decreasing function from zero to one), *i.e.*, $\bar{F}_Z(z) = \underline{F}_Z(z) = F_Z(z)$. An interval expresses only incertitude. Its p-box looks like a rectangular box whose upper and lower bounds jump from zero to one at the endpoints of the interval. A precise scalar number lacks both kinds of uncertainty. Its p-box is just a step from 0 to 1 in correspondence of the scalar value itself **[Ferson et al. 2010]**.

4 In the same way, the limiting upper and lower CDFs $\bar{F}_Z(z)$ and $\underline{F}_Z(z)$ on $Z = Y_1 + Y_2$, are obtained by setting Y_1 to its lower bound $a_1 = 2$ and to its upper bound $b_1 = 7$, respectively, and letting Y_2 vary according to its (aleatory) probability distribution, *i.e.*, $p_{Y_2}(y_2) \sim ln(\mu_2 = 1.6094, \sigma_2 = 0.4214)$. Conversely, the limiting upper and lower CDFs $\bar{F}_Z(z)$ and $\underline{F}_Z(z)$, respectively, on both $Z = Y_2/Y_1$ and $Z = Y_2 - Y_1$ are obtained by setting Y_1 to its upper bound $b_1 = 7$ and to its lower bound $a_1 = 2$, respectively, and letting Y_2 vary according to its (aleatory) probability distribution, *i.e.*, $p_{Y_2}(y_2) \sim ln(\mu_2 = 1.6094, \sigma_2 = 0.4214)$.

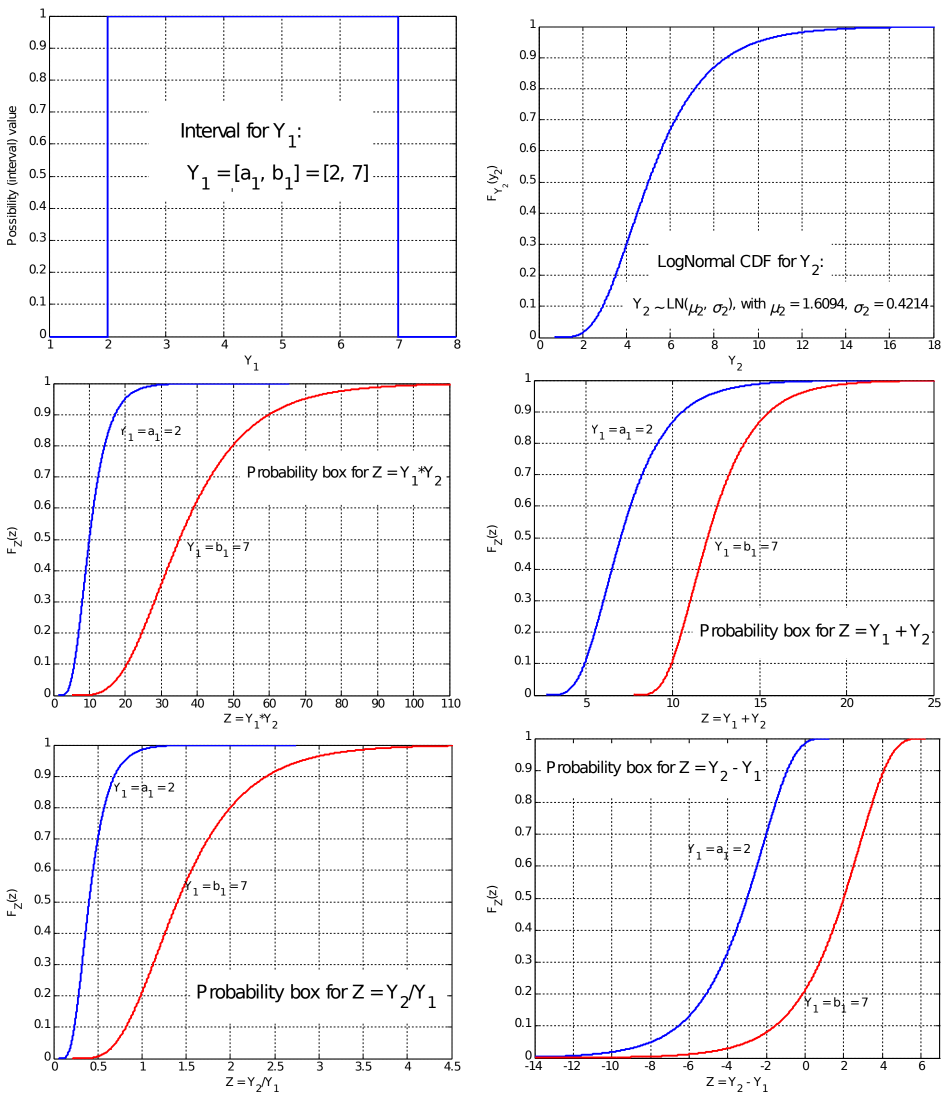

Figure 4.1 – *Top, left: interval* $[a_1, b_1] = [2, 7]$ *representing the (epistemic) uncertainty on variable* Y_1*; top, right: lognormal CDF* $F_{Y_2}(y_2) = ln(\mu_2, \sigma_2)$ *with* $\mu_2 = 1.6094$ *and* $\sigma_2 = 0.4214$ *representing the (aleatory) uncertainty on variable* Y_2*; middle, left: probability box* $[\bar{F}_Z(z), \underline{F}_Z(z)]$ *for the product* $Z = Y_1 \times Y_2$*; middle, right: probability box* $[\bar{F}_Z(z), \underline{F}_Z(z)]$ *for the sum* $Z = Y_1 + Y_2$*; bottom, left: probability box* $[\bar{F}_Z(z), \underline{F}_Z(z)]$ *for the quotient* $Z = Y_2/Y_1$*; bottom, right: probability box* $[\bar{F}_Z(z), \underline{F}_Z(z)]$ *for the difference* $Z = Y_2 - Y_1$

4.3 Discussion

Based on the considerations and examples above, the **advantages of probability bound analysis** can be summarized as follows [Ferson et Hajagos 2004; Ferson et Tucker 2006; Ferson 1999; Ferson et al. 2007, 2010; Ferson et Ginzburg 1996]:

- ▷ it distinguishes *variability* and *incertitude*;
- ▷ it permits analysts to make risk calculations *without* requiring overly *precise assumptions* about parameter values, dependence among variables, or distribution shape. Actually, probability bounds analysis brings together the classically grounded research on bounding probabilities with the recent development of methods for calculation to solve the two fundamental problems in risk assessment of i) not knowing precisely the probability distributions and ii) not knowing exactly the interdependencies among them [Ferson 1996b];
- ▷ it may be especially useful in risk assessments where probabilistic characterizations are desired and *empirical information* is *limited* [Ferson 1999]. Probability bounds analysis does not require a great deal of data, but it can make use of a great deal more of the information that is available to inform a decision than, *e.g.*, interval analysis. Thus, in

general, probability bounds analysis allows to obtain fully rigorous results even when the empirical information is very poor, which is exactly the situation most risk assessors face in their work;

- ▷ it gives the same answer as interval analysis does when only range information is available. It also gives the same answers as Monte Carlo analysis does when information is abundant enough to precisely specify input distributions and their dependencies. Thus, it is a *generalization* of both interval analysis and probability theory;
- ▷ it is *guaranteed* to *bound* answers (and it also puts sure bounds on Monte Carlo results);
- ▷ it produces bounds that get *narrower* with *better empirical information*;
- ▷ it often produces *optimal* solutions;
- ▷ it supports *all* standard mathematical *operations*;
- ▷ it is computationally *faster* than Monte Carlo.

On the contrary, the **disadvantages of probability bound analysis** can be summarized as follows [Ferson et Ginzburg 1996; Ferson 1999; Ferson et al. 2010]:

- ▷ uncertainty *must* be represented by cumulative distribution functions (CDFs) (*i.e.*, probability boxes);
- ▷ probability boxes do not show what is *most likely within* the box (in other words, there are no "shades of gray" or second-order information);
- ▷ although probability boxes are guaranteed to bound answers, they may *not* express the *tightest* possible bounds given available information (in other words, they may *overestimate* epistemic uncertainty as interval analysis does);
- ▷ *infinite tails* of the CDFs must be *truncated* for the ease of computation with probability boxes;
- ▷ optimal bounds may become *expensive* to compute when parameters are *repeated* (see also interval analysis, described in chapter 3).

5

Evidence theory

Evidence theory (also known as Dempster-Shafer theory or theory of belief functions) in the two forms proposed by **[Dempster 1967]** and **[Shafer 1976]** allows for the incorporation and representation of *incomplete* information: its motivation is to be able to treat situations where there is more information than an interval, but *less* than a *single specific probability distribution*. The theory is able to produce epistemic-based uncertainty descriptions and in particular *probability intervals*.

In § 5.1, thorough details about evidence theory are provided; in § 5.2, the issue of uncertainty propagation within evidence theory is treated; finally, in § 5.3, the advantages and disadvantages of evidence theory are thoroughly discussed.

5.1 Uncertainty representation in evidence theory

Fuzzy measures provide powerful mathematical languages for the representation of the epistemic uncertainty in the attribution of an element y to a particular member A of a countable set. For example, suppose that y is a parameter whose values may vary in a given range Y also called *Universe of Discourse* (U_Y): then, the epistemic uncertainty associated to the ambiguity of the value of y can be represented by assigning to each crisp set in Y a value which represents the degree of evidence that y belongs to such set. Thus, fuzzy measures deal with the uncertainty in the assignment of y to crisp sets, which in turn are not uncertain.

It is important to underline that the theory of fuzzy measures is different from the theory of fuzzy sets which deals with the uncertainty associated with vague, linguistic information. In the case of fuzzy set theory, the linguistic statements are represented by overlapping fuzzy sets, thus with no sharp boundaries: correspondingly, due to the vagueness in the available information a given $y \in Y$ may simultaneously belong to several sets with different degrees of membership.

Thus, the difference between a fuzzy measure and a fuzzy set is clear: the former represents the uncertainty in the assignment of an element to a given crisp set, due to lack of knowledge or information deficiency, whereas the latter represents the uncertainty in the definition of the boundaries of a set, due to a lack of sharp boundaries deriving from vague information **[Klir et Yuan 1995]**.

For the formal definition of fuzzy measures, let us consider a finite U_Y and an element $y \in Y$ which is not fully characterized, *i.e.*, it might belong to more than one crisp set in Y. Let $P(Y)$ denote the so called *power set* of Y, *i.e.*, the set of all subsets of Y. For a given set $A \subseteq P(Y)$, the uncertainty in the assignment of y to A is quantitatively represented by the value of a function $g(A)$ which maps to $[0, 1]$ the available evidence regarding the membership of y in A.

Any fuzzy measure satisfies the *minitivity* and *maxitivity* constraints with respect to the conjunction and disjunction of two events A and B:

$$g(A \cap B) \leq \min[g(A), g(B)] \tag{5.1}$$

$$g(A \cup B) \geq \max[g(A), g(B)] \tag{5.2}$$

There are two forms of fuzzy measure functions, namely the *belief measure*, $Bel(A)$, associated to pre-conceived notions, and the *plausibility measure* $Pl(A)$, associated with plausible information.

The belief measure represents the *degree of belief*, based on the available evidence, that a given element of Y belongs to A as well as to any of the subsets of A; it is the degree of belief in set A, based on the available evidence. In this sense, the different subsets of Y may be viewed as the answers to a particular question, some of which are correct but it is not known which ones with full certainty.

A fundamental property of the belief function is that:

$$Bel(A) + Bel(\bar{A}) \leq 1 \tag{5.3}$$

Thus, the specification of the belief function is capable of incorporating a lack of confidence in the occurrence of the event defined by subset A, quantitatively manifested in the sum of the beliefs of the occurrence ($Bel(A)$) and non occurrence ($Bel(\bar{A})$) being less than one.

The difference $1 - Bel(A) + Bel(\bar{A})$ is called *ignorance*. When the ignorance is 0, the available evidence justifies a probabilistic description of the uncertainty.

The plausibility measure can be interpreted as the total evidence that a particular element of Y belongs not only to A or any of its subsets, as for $Bel(A)$, but also to any set which overlaps with A.

A fundamental property of the plausibility function is that:

$$Pl(A) + Pl(\bar{A}) \geq 1 \tag{5.4}$$

Thus, the specification of the plausibility function is capable of incorporating a recognition of alternatives in the occurrence of the event defined by subset A, quantitatively manifested in the sum of the plausibilities of the occurrence ($P(A)$) and non occurrence ($Pl(\bar{A})$) being greater than or equal to one.

The links with the belief measure are:

$$Pl(A) = 1 - Bel(\bar{A}) \tag{5.5}$$

$$Bel(A) = 1 - Pl(\bar{A}) \tag{5.6}$$

from which it follows that

$$Bel(A) \leq Pl(A) \tag{5.7}$$

The representation of uncertainty based on the above two fuzzy measures falls under the framework of *evidence theory* [Shafer 1976]. Whereas in probability theory, a single probability distribution function is introduced to define the probabilities of any event, represented as a subset of the sample space, in evidence theory there are two measures of the likelihood, belief and plausibility. Also, in contrast to the inequalities (5.3) and (5.4), probability theory imposes more restrictive conditions on the specification of likelihood as a result of the requirement that the probabilities of the occurrence and nonoccurrence of an event must sum to one (see (5.22) below).

Evidence theory allows epistemic uncertainty (imprecision) and aleatory uncertainty (variability) to be treated separately within a single framework. Indeed, the belief and plausibility functions provide mathematical tools to process information which is at the same time of random and imprecise nature.

As a further insight, notice that evidence theory is based on the idea of obtaining *degrees of belief* for one question from subjective probabilities for related questions [Shafer 1990]. To illustrate, suppose that a diagnostic model is available to indicate with reliability (*i.e.* probability of providing the correct result) of 0.9 when a given system is failed. Considering a case in which the model does indeed indicate that the system is failed, this fact justifies a 0.9 degree of belief on such event (which is different from the related event of model correctness for which the probability value of 0.9 is available) but only a 0 degree of belief (not a 0.1) on the event that the system is not failed. This latter belief does not mean that it is certain that the system has failed, as a zero probability would: it merely means that the model indication provides no evidence to support the fact that the system is not failed. The pair of values $\{0.9, 0\}$ constitutes a belief function on the propositions "the system is failed" and "the system is not failed".

From the above simple example, one can appreciate how the degrees of belief for one question (has the system failed?) are obtained from probabilities related to another question (is the diagnostic model correct?).

Denoting by A the event that the system is failed and by m the diagnostic indication of the system state, the conditional probability $P(m|A)$, *i.e.* the model reliability, is used as the degree of belief that the system is failed. This is unlike the standard Bayesian analysis, where focus would be on the conditional probability of the failure event given the state diagnosis by the model, $P(A|m)$, which is obtained by updating the prior probability on A, $P(A)$, using Bayes' rule.

As for the interpretation of the measures introduced in evidence theory, **[Shafer 1990]** uses several metaphors for assigning (and hence interpreting) belief functions. The simplest says that the assessor judges that the strength of the evidence indicating that the event A is true, $Bel(A)$, is comparable with the strength of the evidence provided by a witness who has a $Bel(A) \times 100\%$ chance of being reliable. Thus, we have

$$Bel(A) = P(\text{The witness claiming that A is true is reliable}) \tag{5.8}$$

The metaphor is to be interpreted as the diagnostic model analyzed above, witness reliability playing the role of model reliability.

5.1.1 Basic probability assignment

The belief and plausibility functions are defined from a mass distribution $m(A)$ on the sets A of the power set $P(Y)$ of the U_Y, called *basic probability assignment (bpa)*, which expresses the degree of belief that a specific element y belongs to the set A only, and not to any subset of A. The bpa satisfies the following requirements:

$$m: P(Y) \to [0,1] \qquad m(0) = 0 \qquad \sum_{A \in P(Y)} m(A) = 1 \tag{5.9}$$

and defines the belief and plausibility measures as follows,

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{5.10}$$

$$Pl(A) = \sum_{B \cap A \neq 0} m(B) \tag{5.11}$$

Note that from the definition (5.9), it is not required that $m(Y) = 1$, nor that $m(A) \leq m(B)$ when $A \subseteq B$, nor that there be any relationship between $m(A)$ and $m(\bar{A})$. Hence, the bpa is not a fuzzy measure nor a probability distribution.

For each set A of the power set $P(Y)$, the bpa $m(A)$ expresses the proportion to which all available and relevant evidence supports the claim that a particular element y of Y, whose characterization is incomplete, belongs to set A. The value of $m(A)$ pertains solely to set A and does not imply any additional claim regarding subsets of A; if there is additional evidence supporting the claim that the element y belongs to a subset of A, say $B \subseteq A$, it must be expressed by another probability assignment on B, *i.e.* $m(B)$.

Every set $A_i \in P(Y)$ for which $m(A_i) > 0$ is called a *focal element* of m: as the name suggests, focal elements are subsets of Y on which the available evidence focuses. When Y is finite, m can be fully characterized by a list of its focal elements A_i with the corresponding values $m(A_i)$, which together form the *body of evidence* $\{A_i, m(A_i)\}$.

Total ignorance, then, amounts to the following assignment:

$$m(Y) = 1 \qquad m(A_i) = 0, \quad \forall A_i \neq Y \tag{5.12}$$

which means,

$$Bel(Y) = 1; \quad Bel(A_i) = 0, \quad \forall A_i \neq Y \tag{5.13}$$
$$Pl(Y) = 0; \quad Pl(A_i) = 1, \quad \forall A_i \neq 0 \tag{5.14}$$

Note that contrary to probability theory which assigns the probability mass to individual values of y, the theory of evidence makes basic probability assignments $m(A)$ on sets A of the power set $P(Y)$ of the U_Y (the focal sets). On the other hand, evidence theory encompasses probability theory: when focal elements are disjoint sets, both belief and plausibility are probability measures, *i.e.*, $Bel = P = Pl$ for unions of such sets. Thus, all probability distributions may be interpreted as bpa's.

A possible approximate encoding of a continuous probability distribution function p_Y of a real random variable Y into a bpa proceeds as follows **[Baudrit et al. 2006]**:

1. discretize the range of values of Y into n disjoint intervals $]a_i, a_{i+1}]$, $i = 1, 2, \ldots, q$: these are the focal elements;
2. build the mass distribution of the bpa by assigning $m(]a_i, a_{i+1}]) = p(Y \in [a_i, a_{i+1}])$.

In summary:

- ▷ $m(A)$ is the degree of evidence of membership in set A only; it is the amount of likelihood that is associated with A but without any specification of how this likelihood might be apportioned over A: this likelihood might be associated with any subset of A.
- ▷ $Bel(A)$ gathers the imprecise evidence that asserts A; it is the total evidence of membership in set A and all its subsets, which is quantified according to (5.10) as the minimal amount of probability that *must* be assigned to A by summing the pertinent probability masses of the single values in the focal sets: this amount of likelihood cannot move out of A because the summation in (5.10) involves only subsets B of A;
- ▷ $Pl(A)$ gathers the imprecise evidence that does not contradict A; it is the total evidence of membership in set A, all its subsets and all other sets which intersect with A, which is quantified according to (5.11) as the maximal amount of probability that *could* be assigned to A by summing the pertinent probability masses of the single values in the focal sets: this amount of likelihood could move into A from another intersecting set, because the summation in (5.11) involves all sets B which intersect with A.

Then, an expert *believes* that the evidence supporting set A is at least $Bel(A)$ and *possibly as high as* $Pl(A)$.

5.1.2 Aggregation of multiple sources of evidence

Let us consider the common situation in which imprecise evidence is available from more than one source. For simplicity, let us consider two experts whose evidence is expressed in terms of two sets of bpa's, $m_1(A)$, $m_2(A)$ on the focal sets A of the power set $P(Y)$ of Y. Aggregation of this evidence into a joint bpa $m_{12}(A)$ can be obtained by means of *Dempster's rule* **[Dempster 1967]**:

$$m_{12}(A) = \frac{\sum_{B \cap C} m_1(B) m_2(C)}{1 - K} \qquad \forall A \neq 0 \tag{5.15}$$

where the complementary normalization factor K is given by

$$K = \sum_{B \cap C = 0} m_1(B) m_2(C) \tag{5.16}$$

According to (5.15) and (5.16) above, the degree of evidence $m_1(B)$ regarding focal set $B \in P(Y)$, from the first source and the degree of evidence $m_2(C)$ focused on focal set $C \in P(Y)$, from the second source, are aggregated by taking their product $m_1(B) m_2(C)$ focused on the intersection focal set $B \cap C = A$. This way of combining evidence sources is analogous to the way in which in probability theory joint probability density functions (PDFs) are calculated from two independent marginal PDFs and is thus justified on the same grounds. However, some intersections $B \cap C$ of different focal elements B and C, from the first and second source, may result in the same set A so that one must sum their product contribution to obtain $m_{12}(A)$.

Furthermore, some of the intersections may be the empty set, for which $m_{12}(0) = 0$. Then, the sum of products $m_1(B)m_2(C)$ of all focal elements B of m_1 and C of m_2 such that $B \cap C \neq 0$ is equal to $1 - K$, so that a normalized joint basic assignment m_{12} (as required by (5.9)) is obtained by dividing by K given in (5.16).

5.1.3 Relation to probability measures

Let us consider a bpa only on individual values (singletons) $y \in Y$ but not on any other subset A of the power set $P(Y)$, *i.e.*, $m(y) = Bel(y), y \in Y, m(A) = 0, \forall A \in Y$. Then, $m(y)$ is a probability measure, commonly denoted as $p(y)$, which maps the evidence on singletons to the unit interval [0, 1].

It is then clear that the key distinction between a probability measure and either a belief or plausibility measure is that in the former all evidence is focused on singletons y only whereas in belief and plausibility measures the evidence is focused on (focal) subsets A of the power set $P(Y)$.

Obviously, from the probability measure $p(y)$ defined on all singletons $y \in Y$ one can compute the probability measure $p(A)$ of any set A, which is simply a collection of singletons:

$$p(A) = \sum_{y \in A} p(y), \quad \forall A \in P(Y) \tag{5.17}$$

Notice that in this case in which the basic probability assignment focuses only on singletons $y \in Y$, as required for probability measures, the belief, plausibility and probability of a set A are all equal:

$$Bel(A) = Pl(A) = p(A) = \sum_{y \in A} p(y) = \sum_{y \in A} m(y) \quad \forall A \in P(Y) \tag{5.18}$$

Thus, belief and plausibility measures overlap when all the evidence is focused only on singletons $y \in Y$ and they both become probability measures.

Also, considering for simplicity only two focal sets A and B, a probability measure arises if:

$$Bel(A \cup B) = Bel(A) + Bel(B) \qquad A \cap B = 0 \tag{5.19}$$

$$Pl(A \cup B) = Pl(A) + Pl(B) \qquad A \cap B = 0 \tag{5.20}$$

On the contrary, when evidence does not reside exclusively on the singletons $y \in Y$, it can be shown that

$$Bel(A) \leq p(A) \leq Pl(A) \tag{5.21}$$

Thus, the dual measures of belief and plausibility form intervals $[Bel(A), Pl(A)] \forall A \in P(X)$ which can be viewed as imprecise estimates of probabilities derived from the coarse evidence expressed by the basic probability assignment.

Finally, from (5.4), (5.5) and (5.18) it follows that

$$p(A) + p(\bar{A}) = 1 \tag{5.22}$$

which imposes a more stringent condition on the probability measure than (5.3) and (5.4) do on the belief and plausibility measures, respectively.

5.2 Uncertainty propagation in evidence theory

Referring to the uncertainty propagation task framed in the introduction, let a model whose output is a function $Z = f(\mathbf{Y}) = f(Y_1, Y_2, \ldots, Y_j, \ldots, Y_n)$ of n uncertain variables $Y_j, j = 1, 2, \ldots, n$, whose uncertainty is described by the so-called *body of evidence* or *basic probability assignment* (bpa), *i.e.*, by a list of discrete (focal) sets

$$\left\{A_{Y_1,t_1}, A_{Y_2,t_2}, \ldots, A_{Y_j,t_j}, \ldots, A_{Y_n,t_n}\right\} = \left\{[\underline{y}_{1,t_1}, \bar{y}_{1,t_1}], [\underline{y}_{2,t_2}, \bar{y}_{2,t_2}], \ldots, [\underline{y}_{j,t_j}, \bar{y}_{j,t_j}], \ldots, [\underline{y}_{n,t_n}, \bar{y}_{n,t_n}]\right\}$$
$$t_j = 1, 2, \ldots, q_j, j = 1, 2, \ldots, n$$

and by the corresponding probability masses $\{m_{Y_1,t_1}, m_{Y_2,t_2}, \ldots, m_{Y_j,t_j}, \ldots, m_{Y_n,t_n}\}, t_j = 1, 2, \ldots, q_j, j = 1, 2, \ldots, n$. In summary, the body of evidence (or basic probability assignment) $\{(A_{Z,t_Z}, m_{A_Z,t_Z}) : t_Z = 1, 2, \ldots, q_Z\}$ for the output $Z = f(\mathbf{Y}) = f(Y_1, Y_2, \ldots, Y_j, \ldots, Y_n)$ is computed by constructing the Cartesian product of the n collections of the interval-mass pairs $\{(A_{Y_j,t_j}, m_{Y_j,t_j}) : t_j = 1, 2, \ldots, q_j, j = 1, 2, \ldots, n\}$. In more detail, the discrete focal sets $A_{Z,t_Z}, t_z = 1, 2, \ldots, q_z$, are obtained by evaluating $A_{Z,t_Z} = f(A_{Y_1,t_1}, A_{Y_2,t_2}, \ldots, A_{Y_j,t_j}, \ldots, A_{Y_n,t_n})$ for *all possible combinations* of $t_1 = 1, 2, \ldots, q_1, t_2 = 1, 2, \ldots, q_2, \ldots, t_j = 1, 2, \ldots, q_j, \ldots, t_n = 1, 2, \ldots, q_n$ (notice that by so doing $q_z = q_1 \cdot q_2 \cdot \ldots \cdot q_j \cdot \ldots \cdot q_n$). Assuming that the uncertain variables $Y_j, j = 1, 2, \ldots, n$, are *independent*, the corresponding probability masses $m_{A_{Z,t_Z}}, t_z = 1, 2, \ldots, q_z = q_1 \cdot q_2 \cdot \ldots \cdot q_j \cdot \ldots \cdot q_n$, are then simply obtained as the *product* of the probability masses of the focal sets of the uncertain input variables $Y_j, j = 1, 2, \ldots, n$, *i.e.*, $m_{A_{Z,t_Z}} = m_{A_Z} = f(A_{Y_1,t_1}, A_{Y_2,t_2}, \ldots, A_{Y_j,t_j}, \ldots, A_{Y_n,t_n}), t_Z = m_{Y_1,t_1} \cdot m_{Y_2}, t_2 \cdot \ldots \cdot m_{Y_j,t_j} \cdot \ldots \cdot m_{Y_n}, t_n$ for *all possible combinations* of $t_1 = 1, 2, \ldots, q_1, t_2 = 1, 2, \ldots, q_2, \ldots, t_j = 1, 2, \ldots, q_j, \ldots, t_n = 1, 2, \ldots, q_n$[5]. Finally, the plausibility $Pl(A)$ and belief $Bel(A)$ for each set A of interest contained in the universe of discourse U_Z of Z can be obtained as $Pl(A) = \sum_{A_{Z,t_Z} \cap A \neq 0} m_{A_Z,t_Z}$ and $Bel(A) = \sum_{A_{Z,t_Z} \subseteq A} m_{A_Z,t_Z}$, respectively **[Ferson et al. 2003, 2004; Helton et al. 2007, 2008; Sentz et Ferson 2002; Baudrit et Dubois 2005; Baudrit et al. 2003; Fetz 2001; Fetz et Oberguggenberger 2004; Helton et Oberkampf 2004; Moral et Wilson 1996; Oberkampf et Helton 2002; Oberkampf et al. 2001; Tonon 2004; Tonon et al. 2000a,b]**.

A body of evidence

By way of example, let Y_1 be represented by the body of evidence (or basic probability assignment)

$$\begin{aligned}\left\{(A_{Y_1,t_1}, m_{Y_1,t_1}) \quad t_1 = 1, 2, q_1 = 3\right\} = \{&(A_{Y_1,1} = [1, 3], m_{Y_1,1} = 0.2),\\ &(A_{Y_1,2} = [3, 5], m_{Y_1,2} = 0.5),\\ &(A_{Y_1,3} = [4, 6], m_{Y_1,3} = 0.3)\}\end{aligned}$$

and Y_2 be represented by the body of evidence (or basic probability assignment)

$$\begin{aligned}\left\{(A_{Y_2,t_2}, m_{Y_2,t_2}) : t_2 = 1, 2, q_2 = 3\right\} = \{&(A_{Y_2,1} = [1, 4], m_{Y_2,1} = 0.4),\\ &(A_{Y_2,2} = [2, 6], m_{Y_2,2} = 0.1),\\ &(A_{Y_2,3} = [4, 8], m_{Y_2,3} = 0.5)\}\end{aligned}$$

In addition, for clarity figure 5.1, top, left and right shows the corresponding upper and lower CDFs $\bar{F}_{Y_1}(y_1) = Pl_{Y_1}(-\infty, y_1), \bar{F}_{Y_2}(y_2) = Pl_{Y_2}(-\infty, y_2), \underline{F}_{Y_1}(y_1) = Bel_{Y_1}(-\infty, y_1), \underline{F}_{Y_2}(y_2) = Bel_{Y_2}(-\infty, y_2)$ of Y_1 and Y_2, respectively.

Table 5.1 reports the body of evidence (or basic probability assignment) $\left\{(A_{Z,t_Z}, m_{A_Z,t_Z}) : t_Z = 1, 2, \ldots, q_Z = q_1 \cdot q_2 = 9\right\}$ of the output $Z = Y_1 + Y_2$ constructed (under the assumption of independence between Y_1 and Y_2) by means of the Cartesian product of the two collections of interval-mass pairs $\left\{(A_{Y_1,t_1}, m_{Y_1,t_1}) : t_1 = 1, 2, q_1 = 3\right\}$ and $\{(A_{Y_2,t_2}, m_{Y_2,t_2}) : t_2 = 1, 2, q_2 = 3\}$ indicated above. Again, for clarity the upper and lower CDFs of Z, *i.e.*, $\bar{F}_Z(z) = Pl_Z(-\infty, z)$ and $\underline{F}_Z(z) = Bel_Z(-\infty, z)$, respectively, are shown in figure 5.1, bottom.

[5] It is of paramount importance to note that if the uncertain variables $Y_j, j = 1, 2, \ldots, n$ are not independent, completely different approaches have to be undertaken to calculate $m_{A_Z,t_Z}, t_z = 1, 2, \ldots, q_z$. The interested reader is referred to, *e.g.*, **[Ferson et al. 2004]** for further details.

$Z = Y_1 + Y_2$	$(A_{Y_1,1} = [1,3], m_{Y_1,1} = 0.2)$	$(A_{Y_1,2} = [3,5], m_{Y_1,2} = 0.5)$	$(A_{Y_1,3} = [4,6], m_{Y_1,3} = 0.3)$
$(A_{Y_2,1} = [1,4], m_{Y_2,1} = 0.4)$	$(A_{Z,1} = [2,7], m_{Z,1} = 0.08)$	$(A_{Z,2} = [4,9], m_{Z,2} = 0.2)$	$(A_{Z,3} = [5,10], m_{Z,3} = 0.12)$
$(A_{Y_2,2} = [2,6], m_{Y_2,2} = 0.1)$	$(A_{Z,4} = [3,9], m_{Z,4} = 0.02)$	$(A_{Z,5} = [5,11], m_{Z,5} = 0.05)$	$(A_{Z,6} = [6,12], m_{Z,6} = 0.03)$
$(A_{Y_2,3} = [4,8], m_{Y_2,3} = 0.5)$	$(A_{Z,7} = [5,11], m_{Z,7} = 0.1)$	$(A_{Z,8} = [7,13], m_{Z,8} = 0.25)$	$(A_{Z,9} = [8,14], m_{Z,9} = 0.15)$

Table 5.1 – *Body of evidence (or basic probability assignment)* $\{(A_{Z,t_Z}, m_{A_Z,t_Z}): t_Z = 1,2,\dots,q_Z = q_1 \cdot q_2 = 9\}$ *for the output* $Z = Y_1 + Y_2$ *obtained by means of the Cartesian product between the interval-mass pairs* $\{(A_{Y_1,1} = [1,3], m_{Y_1,1} = 0.2), (A_{Y_1,2} = [3,5], m_{Y_1,2} = 0.5), (A_{Y_1,3} = [4,6], m_{Y_1,3} = 0.3)\}$ *of* Y_1 *and* $\{(A_{Y_2,1} = [1,4], m_{Y_2,1} = 0.4), (A_{Y_2,2} = [2,6], m_{Y_2,2} = 0.1), (A_{Y_2,3} = [4,8], m_{Y_2,3} = 0.5)\}$ *of* Y_2

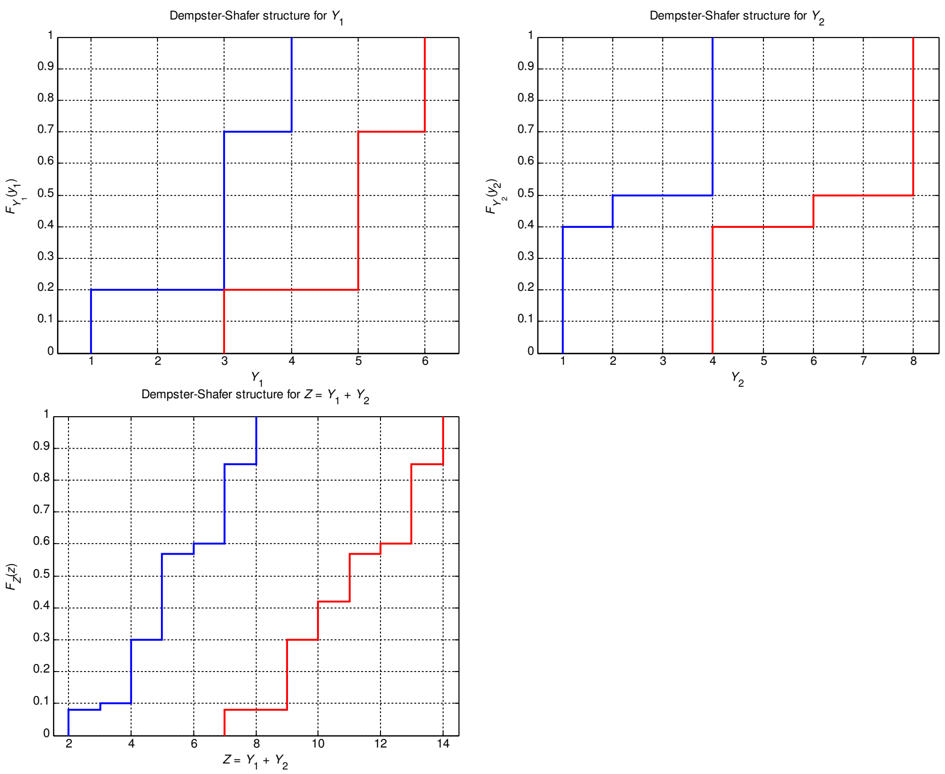

Figure 5.1 – *Top, left: upper and lower CDFs of* Y_1, *i.e.,* $\bar{F}_{Y_1}(y_1) = Pl_{Y_1}(-\infty, y_1)$ *and* $\underline{F}_{Y_1}(y_1) = Bel_{Y_1}(-\infty, y_1)$, *respectively, associated to the bpa* $\{(A_{Y_1,t_1}, m_{Y_1,t_1}): t_1 = 1,2, q_1 = 3\}$ *reported in table 5.1; top, right: upper and lower CDFs of* Y_2, *i.e.,* $\bar{F}_{Y_2}(y_2) = Pl_{Y_2}(-\infty, y_2)$ *and* $\underline{F}_{Y_2}(y_2) = Bel_{Y_2}(-\infty, y_2)$, *respectively, associated to the bpa* $\{(A_{Y_2}, t_2, m_{Y_2,t_2}): t_2 = 1,2, q_2 = 3\}$ *reported in table 5.1; bottom: upper and lower CDFs of* $Z = Y_1 + Y_2$, *i.e.,* $\bar{F}_Z(z) = Pl_Z(-\infty, z)$ *and* $\underline{F}_Z(z) = Bel_Z(-\infty, z)$, *respectively, associated to the bpa* $\{(A_{Z,t_Z}, m_{A_Z,t_Z}): t_Z = 1,2,\dots,q_Z = q_1 \cdot q_2 = 9\}$ *reported in table 5.1.*

5.3 Discussion

Based on the considerations and examples above, the **advantages of evidence theory** can be summarized as follows [Ferson 1999; Ferson et al. 2003, 2004; Helton et al. 2007, 2008; Sentz et Ferson 2002]:

- ▷ it distinguishes *variability* and *incertitude*;
- ▷ it permits analysts to make risk calculations *without* requiring overly *precise assumptions* about parameter values, dependence among variables, or distribution shape [Ferson 1996b];
- ▷ it may be especially useful in risk assessments where probabilistic characterizations are desired and *empirical information* is *limited* [Ferson 1999];
- ▷ basic probability assignments can be constructed with (almost) *any* kind of data (*e.g.*, data sets in which measurement error about the values is expressed as intervals, censored data, datasets in which observed values have intrinsic indistinguishability, ...);
- ▷ it encompasses probability theory (*i.e.*, the two theories coincide when the basic probability assignments are singletons or disjoint intervals); it encompasses interval analysis (the two theories coincide when the basic probability assignment is constituted by a *single* interval): as a consequence, it includes also probability bound analysis; finally, notice that it encompasses also possibility theory (see the following chapter);
- ▷ it produces bounds that get *narrower* with *better empirical information*;
- ▷ it supports *all* standard mathematical *operations*;
- ▷ it is simple to implement.

On the contrary, the **disadvantages of evidence theory** can be summarized as follows [Ferson et al. 2003, 2004; Helton et al. 2007, 2008; Sentz et Ferson 2002; Ferson et al. 2010]:

- ▷ it is *not* yet *widely known* and *applied* so that it has not been broadly accepted in the risk assessment community. Much effort has been made in this area, often with a mathematical orientation, but no convincing framework for risk assessment in practice presently exists based on these alternative theories. Further research is required to make this theory operational in a risk assessment context;
- ▷ basic probability assignments (*i.e.*, intervals with the associated probabilities) do not show what is *most likely within* the bpa's (in other words, there are no shades of gray or second-order information within the bpa's);
- ▷ Dempster's rule is weird and controversial.

The next chapter describes possibility theory, which is a subset of evidence theory.

6

Possibility theory

The rationale for using possibility distributions to describe epistemic uncertainty lies in the fact that a possibility distribution defines a *family* of probability distributions (bounded above and below by the so-called possibility and necessity functions, respectively), thus it allows to account for the expert's inability to specify a *single* probability distribution [Baudrit et Dubois 2006; Baudrit et al. 2006, 2008; Dubois 2006; Dubois et Prade 1988]. Notice that possibility theory can be considered a special case of evidence theory (see previous chapter).

In § 6.1, thorough details about possibility theory are provided; in § 6.2, the issue of uncertainty propagation within possibility theory is treated; finally, in § 6.3, the advantages and disadvantages of possibility theory are thoroughly discussed.

6.1 Uncertainty representation using possibility theory

Possibility theory is a special branch of evidence theory that deals with bodies of evidence whose focal elements $A_1, A_2, \ldots, A_n$ on the power set $P(Y)$ of the UOD U_Y (also called Y for brevity) are *nested*, *i.e.*

$$A_1 \subset A_2 \subset \ldots \subset A_n \in P(Y) \tag{6.1}$$

Then, the belief and plausibility measures $Bel(A_i)$ and $Pl(A_i)$ are said to represent a consonant body of evidence in the sense that the evidence allocated to the various subsets does not conflict. For a consonant body of evidence,

$$Bel(A \cap B) = \min\left[Bel(A), Bel(B)\right] \tag{6.2}$$

$$Pl(A \cup B) = \max\left[Pl(A), Pl(B)\right] \tag{6.3}$$

for any pairs of focal sets $A, B \in P(Y)$.

Comparing (6.2) and (6.3) with the general properties of fuzzy measures (5.1) and (5.2), one can see that possibility theory is based on the extreme values of fuzzy measures with respect to intersection and union sets. On the other hand, comparing to (5.17)–(5.20) one can see that possibility theory is *minitive* and *maxitive*, and not additive as is probability theory.

Consonant belief and plausibility measures are referred to as *necessity*, N, and *possibility*, Π, measures, respectively, characterized by the properties (6.2) & (6.3) and with the duality relationships (5.3) and (5.4) still holding. In addition, necessity and possibility measures constrain each other in a strong way:

$$N(A) > 0 \Rightarrow \Pi(A) = 1 \tag{6.4}$$

$$\Pi(A) < 1 \Rightarrow N(A) = 0 \tag{6.5}$$

Necessity and possibility measures are sometimes defined axiomatically by equations (6.2) and (6.3), from which their structure can be derived as theorems.

6.1.1 Numerical possibility theory

The basic notion of numerical possibility theory is the *possibility distribution* which assigns to each value y in a range Y (also called Universe of Discourse, U_Y) a degree of possibility $\pi_Y(y) \in [0, 1]$ of being the correct value of an uncertain variable (not necessarily random) **[Dubois 2006]**. In a sense, then, the possibility distribution $\pi_Y(y)$ reflects the degree of similarity between y and an ideal prototype (the true value) for which the possibility degree is 1. Thus, $\pi_Y(y)$ measures the distance between y and the prototype: in some cases, this may be determined objectively, *e.g.* by a defined measurement procedure **[Krishnapuram et Keller 1993]**; in other cases, it may be based on subjective judgment. When $\pi_Y(y) = 0$ for some y, it means that the outcome y is considered an impossible situation. When $\pi_Y(y) = 1$ for some y, it means that the outcome y is possible, *i.e.*, is just unsurprising, normal, usual **[Dubois 2006]**. This is a much weaker statement than when probability is 1.

If y and y' are such that $\pi_Y(y) > \pi_Y(y')$, then y is considered to be a more plausible value than y', *i.e.*, closer to the prototype. A possibility distribution is thus an upper, semi-continuous mapping from the real line to the unit interval describing what an analyst knows about the more or less plausible values y of the uncertain variable ranging on Y. These values are assumed to be mutually exclusive, since the uncertain variable takes on only one value, the true one. This also gives the normalization condition:

$$\exists y\colon\ \pi_Y(y) = 1 \tag{6.6}$$

Since the condition $\pi_Y(y) = 1$ means that the fact that the uncertain variable is equal to x is just unsurprising, normal, usual (a much weaker statement than $p_Y(y) = 1$), the normalization (6.6) claims that at least one value is viewed as totally possible. Indeed, if $\forall y \in Y, \pi_Y(y) < 1$, the uncertainty representation given by the possibility distribution would be logically inconsistent since it would suggest that all values in Y are only partially possible. In this respect, the degree of consistency of a sub-normalized possibility distribution ($\sup_{y \in Y} \pi_Y(y) < 1$) is:

$$cons(\pi_Y) = \sup_{y \in Y} \pi_Y(y) \tag{6.7}$$

A possibility distribution π_1 is at least as informative (*specific*) as another one π_2 if and only if $\pi_1 \leq \pi_2$ **[Yager 1992]**. In the particular case that $\forall y \in Y, \pi_Y(y) = 1, \pi_Y(y)$ contains no information since it simply describes the fact that any value $y \in Y$ is possible; in this case the possibility measure is said to be *vacuous*.

6.1.2 Relationship between possibility distribution and possibility and necessity measures

Every possibility and necessity measures $\Pi(A), N(A), \quad \forall A \in P(Y)$ are uniquely represented by the associated possibility distribution $\pi_Y(y)$ through the following maximization and minimization relationships, respectively:

$$\Pi(A) = \sup_{y \in A} \pi_Y(y) \tag{6.8}$$

$$N(A) = 1 - \Pi(\bar{A}) = \inf_{y \notin A} (1 - \pi_Y(y)) \tag{6.9}$$

6.1.3 Relationship between possibility distribution and fuzzy sets

Possibility theory can be formulated not only with respect to nested bodies of evidence (6.1) but also in relations to fuzzy sets. Indeed, fuzzy sets are also based on families of nested sets, the so-called α-cuts **[Zadeh 1965]**. Consider a fuzzy set F on the range Y (or UOD U_Y). Given $y \in Y$ (or $y \in U_Y$), the membership function value $\mu_{F,Y}(y)$ represents the *degree of compatibility* of the value y with the linguistic concept expressed by F. On the other hand, given the proposition "$\mathcal{Y}$ *is* y" on the linguistic variable $\mathcal{Y}$, it is more meaningful to interpret $\mu_{F,Y}(y)$ as the *degree of possibility* that the linguistic variable $\mathcal{Y}$ is equal to y. With this interpretation, the possibility $\mu_{F,Y}(y)$ of "$\mathcal{Y} = y$" is numerically equal to the degree $\mu_{F,Y}(y)$ with which y belongs to F, *i.e.* is compatible with the linguistic concept expressed by it:

$$\pi_{F,Y}(y) = \mu_{F,Y}(y) \quad \forall y \in U_Y \tag{6.10}$$

Then, given $\mu_{F,Y}(y)$, the associated possibility measure Π_F is defined for all $A \in P(Y)$ as:

$$\Pi_F(A) = \max_{y \in A} \pi_{F,Y}(y) \tag{6.11}$$

This measure expresses the uncertainty regarding the actual value of variable Y given by the proposition "Y *is* y".

Dually, for normal fuzzy sets the associated necessity measure can be computed as:

$$N_F(A) = 1 - \Pi_F(\bar{A}) \quad \forall A \in P(Y) \tag{6.12}$$

Thus, the possibility and necessity measures can be seen to be formally equivalent to fuzzy sets, the membership grade of an element y to the fuzzy set corresponding to the degree of possibility of the singleton.

A problem arising from this equivalence is that the intersection of two consonant belief functions, like the possibility and necessity measures, using Dempster's rule is not guaranteed to lead to a consonant result. There exist methods to combine consonant possibility measures to obtain a consonant possibility measure, but they are rather cumbersome **[Klir et Yuan 1995]**.

6.1.4 Possibility and necessity as probability bounds

A unimodal numerical possibility distribution may thus also be viewed as a set of nested confidence intervals, which are the α-cuts $A_\alpha^Y = \left[\underline{y}_\alpha, \bar{y}_\alpha\right] = \{y, \pi_Y(y) \geq \alpha\}$ of $\pi_Y(y)$. The degree of certainty that $A_\alpha^Y = [\underline{y}_\alpha, \bar{y}_\alpha]$ contains the value of the uncertain variable is $N([\underline{y}_\alpha, \bar{y}_\alpha])$, which is equal to 1-α if $\pi_Y(y)$ is continuous. The range of values is widest at possibility level α = 0. Just above possibility level α = 0 is the range of values that are 'just possible' or only 'conceivable'. This interval is the range that everyone would agree contains the true value. It is the most conservative estimate of the uncertainty about a quantity. In contrast, the range of values is narrowest at possibility level α = 1. Thus, this level corresponds to the greatest optimism about the uncertainty. It is the range of values that are identified as 'entirely possible'. This range might in fact be a point, in which case it would be the best estimate of the value. But if it is an interval, it is the narrowest one that would be given by a very confident empiricist (see also § 6.1.1). At intermediate levels 0 < α < 1 are ranges which are intermediate in terms of their possibility.

Conversely, the body of evidence $[(A_1, \lambda_1), (A_2, \lambda_2), \ldots, (A_m, \lambda_m)]$ formed by a nested set of intervals A_i, where $A_i \subset A_{i+1}, i = 1, 2, \ldots, m-1$, with degrees of necessity (*i.e.* certainty) $\lambda_i = N(A_i)$ that A_i contains the value of the uncertain variable ($\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$, due to the monotonicity of the necessity function N) is equivalent to the least informative (specific) possibility distribution that obeys the constraints $\lambda_i = N(A_i), i = 1, 2, \ldots, m$ **[Dubois et Prade 1992]**:

$$\pi_Y(y) = \begin{cases} 1 & \text{if } y \in A_1 \\ \min\limits_{i:\ y \notin A_i} 1 - \lambda_i & \text{otherwise} \end{cases} \tag{6.13}$$

This solution is the least committed one with respect to the available data, since by allowing the greatest possibility degrees in agreement with the constraints it defines the least restrictive possibility distribution. The set of possibility values $\pi_Y(y), y \in Y$ thereby obtained is finite.

Dually, the family of nested confidence intervals $[(A_1, \lambda_1), (A_2, \lambda_2), \ldots, (A_m, \lambda_m)]$ can be reconstructed from the possibility distribution $\pi_Y(y)$ **[Dubois 2006]**).

Under this view, a pair (A, λ) supplied by an expert is interpreted as stating that the *subjective probability* $p_Y(A)$ is at least equal to λ **[Dubois et Prade 1992]**. In particular, the α-cut of a continuous possibility distribution can be interpreted as the inequality

$$P(\text{uncertain variable} \in \lfloor \underline{y}_\alpha, \bar{y}_\alpha \rfloor) \geq 1 - \alpha$$

or equivalently as

$$p(\text{uncertain variable} \notin \lfloor \underline{y}_{\alpha}, \bar{y}_{\alpha} \rfloor) \leq \alpha$$

Thus, we can interpret any pair of dual necessity/possibility functions as lower and upper probabilities induced from specific probability families: degrees of necessity are equated to lower probability bounds and degrees of possibility to upper probability bounds.

Formally, let π_Y be a possibility distribution inducing a pair of necessity/possibility functions $[N, \Pi]$. The probability family $\mathcal{P}(\pi_Y)$ is defined as:

$$\mathcal{P}(\pi_Y) = \{p_Y, \forall A \text{ measurable: } N(A) \leq p_Y(A)\} = \{p_Y, \forall A \text{ measurable: } p_Y(A) \leq \pi_Y(A)\} \tag{6.14}$$

In this case, the probability family $\mathcal{P}(\pi_Y)$ is entirely determined by the probability intervals it generates.

$$\sup_{p_Y} p_Y(A) = \Pi(A) \tag{6.15}$$

$$\inf_{p_Y} p_Y(A) = N(A) \tag{6.16}$$

Similarly, suppose pairs (A_i, λ_i) are supplied by an expert as subjective probabilities that $p(A_i)$ is at least equal to λ_i, where A_i is a measurable set. The probability family $\mathcal{P}(\pi_Y)$ is defined as:

$$\mathcal{P}(\pi_Y) = \{p_Y, \forall A_i \colon \lambda_i \leq p_Y(A_i)\} \tag{6.17}$$

We thus know that $p_Y \in \left\lfloor \underline{p}_Y, \bar{p}_Y \right\rfloor$, with $\bar{p}_Y = \Pi$ and $\underline{p}_Y = N$ **[Dubois et Prade 1992]**.

Notice that possibility, necessity and probability measures never coincide, except for the special case of *perfect evidence*, *i.e.* all the body of evidence is focused on just one singleton whereas there is zero evidence on all other focal elements. The probability and possibility distribution functions are equal and such that one element of the UOD is assigned the value of 1 and all other elements are assigned a value of 0.

As an example **[Klir et Yuan 1995]**, consider a temperature variable $\mathcal{Y}$ taking only integer values. The information about its value is given in terms of the fuzzy proposition $\mathcal{Y}$ is around 21°C as expressed by the fuzzy set F given in figure 6.1, top. The ambiguous information represented by fuzzy set F induces a possibility distribution $\pi_{F,Y}$ that, according to (6.10) coincides with $\mu_{F,Y}$ (*cf.* table 6.1). The (nested) α-cuts of $\mu_{F,Y}$ (figure 6.1, bottom) constitute the focal elements of the corresponding possibilistic body of evidence, whose possibility and necessity measures and basic probability assignments are reported in table 6.1.

Set	Π	N	m
$A_1 = \{21\}$	1	1/3	1/3
$A_2 = \{20, 21, 22\}$	1	2/3	1/3
$A_3 = \{19, 20, 21, 22, 23\}$	1	1	1/3

Table 6.1 – *Possibility, necessity measures and basic probability assignments of the focal elements of the example of figure 6.1*

As a further example, consider the following **[Anoop et Rao 2008; Baraldi et Zio 2008]**. We consider an uncertain parameter y; based on its definition we know that the parameter can take values in the range [4, 6] and the most likely value is 5: to represent this information a triangular possibility distribution on the interval [4, 6] is used, with maximum value at 5; *cf.* figure 6.2.

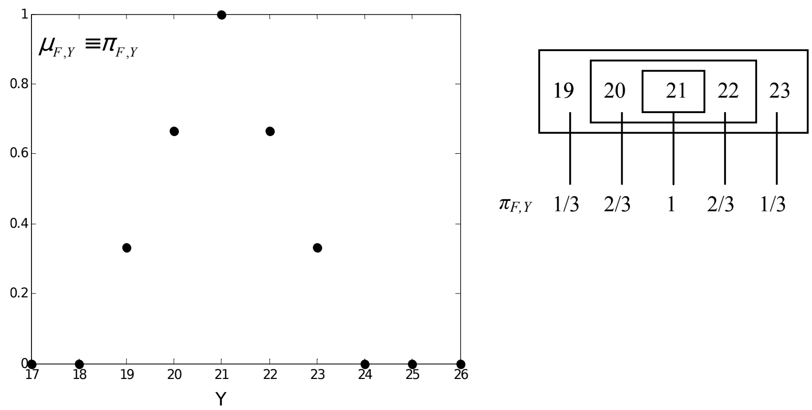

Figure 6.1 – *Possibility distribution of the fuzzy proposition '$\mathcal{Y}$ is around 21°C' (left) and corresponding nested α-cuts of $\mu_{F,Y}$ (right)*

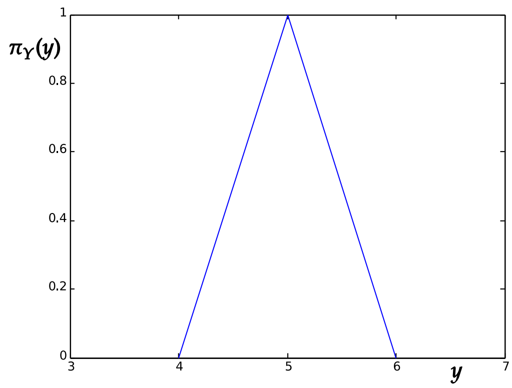

Figure 6.2 – *Possibility function for a parameter uncertain on the interval [4, 6], with maximum value at 5*

Given the possibility function in figure 6.2, we can define different α-cut sets $A^Y_\alpha = \{y: \pi_Y(y) \geq \alpha\}$, for $0 \leq \alpha \leq 1$. For example $A^Y_{0.5} = [4.5, 5.5]$ is the set of y values for which the possibility function is greater than or equal to 0.5. From the triangular possibility distribution in figure 6.2, we can conclude that if A expresses that the parameter lies in the interval [4.5, 5.5], then $0.5 \leq P(A) \leq 1$.

From (6.9) and (6.8) we can deduce the associated cumulative necessity/possibility measures $N(-\infty, y)$ and $\Pi(-\infty, y)$ as shown in figure 6.3. These measures are interpreted as the lower and upper limiting cumulative probability distributions for the uncertain parameter y.

These bounds can be interpreted as for the interval probabilities: the interval bounds are those obtained by the analyst as he/she is not able or willing to precisely assign his/her probability — the interval is the best he/she can do given the information available. However, this makes it essential to understand what information is in fact represented by the possibility function. It is referred to **[Dubois et al. 1993]** and **[Flage et al. 2010b, 2013]** who transform possibility functions to probability distributions and vice-versa.

Finally, a possible approximate encoding of a continuous possibility distribution function $\pi_Y(y)$ of a possibilistic variable Y into a bpa proceeds as follows **[Baudrit et al. 2006]**:

1. determine q (nested) focal elements for Y as the α-cuts; $A^Y_{\alpha_j} = [\underline{y}_{\alpha_j}, \bar{y}_{\alpha_j}], j = 1, 2, \ldots, q$ with $\alpha_0 = \alpha_1 = 1 > \alpha_2 > \ldots > \alpha_q > \alpha_{q+1} = 0$.
2. build the mass distribution of the bpa by assigning $m(A^Y_{\alpha_j}) = \alpha_j - \alpha_{j+1}$ (figure 6.4).

By so doing, we obtain a lower approximation of $\pi_Y(y)$; an upper approximation can be dually obtained.

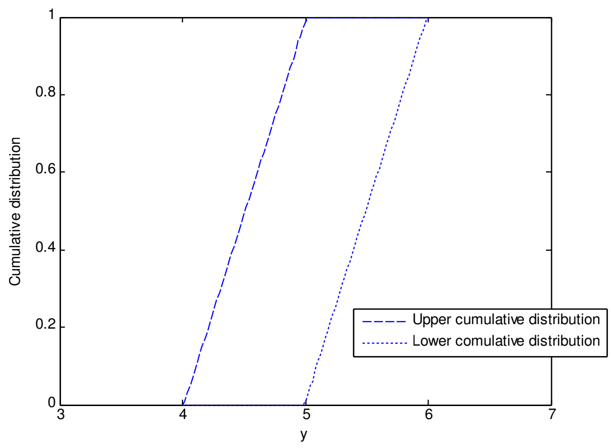

Figure 6.3 – *Bounds for the probability measures based on the possibility function in figure 6.2*

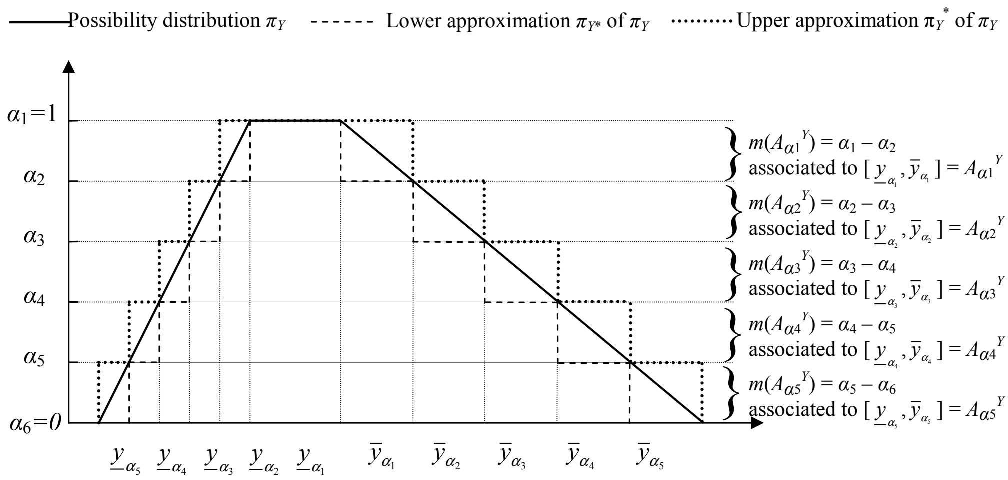

Figure 6.4 – *Approximate encoding of a possibility distribution into a bpa*

6.1.5 A qualitative comparison of possibility and probability theories

As illustrated above, possibility theory is based on a pair of dual measures, possibility and necessity, which are special versions of belief and plausibility measures from evidence theory. On the other hand, probability theory is that subarea in which belief and plausibility measures coincide. This is due to a fundamental difference in the structure of the respective bodies of evidence: families of nested sets for the possibilistic one; singletons for the probabilistic one. Different normalization requirements on possibility and probability distributions then follow: for possibilities, the largest values are required to be 1; for probabilities, their values are required to add to 1. These differences in mathematical properties make each theory suitable for modeling certain types of uncertainty.

A fundamental difference between the possibility and probability theories is their representation of *total ignorance*. In possibility theory, as in evidence theory, total ignorance is represented by the basic probability assignment (5.12) which is equivalent to a unitary possibility distribution on the entire UOD U_Y, *i.e.* $\pi_Y(y) = 1 \; \forall y \in U_Y$. In probability theory, total ignorance is represented by a uniform distribution on the entire UOD U_Y, *i.e.* $p_Y(y) = \frac{1}{|U_Y|} \; \forall y \in U_Y$. This is derived from the fact that in probability theory uncertainty is described by a single probability distribution. This approach appeals to *Laplace's principle of insufficient reason* according to which all that is equally plausible is equally probable and is also justified on the basis of the *maximum entropy approach* **[Gzyl 1995]**. It can however be criticized on the grounds that adopting uniform probabilities to express ignorance implies that the degrees of probability depend on the size $|U_Y|$ of the UOD U_Y and that if no information is available to characterize the uncertain situation under study, then no distribution can be supported: total ignorance should then be expressed in terms of the full set of possible probability distributions on the UOD U_Y so that the probability of a value $y \in U_Y$ is allowed to take any value in [0,1].

As explained above, possibility distributions π_Y and mass distributions (bpa's) m encode probability families and thus allow to represent incomplete probabilistic knowledge. The intervals $[N, \Pi]$ induced from π_Y and $[Bel, Pl]$ induced from m thus provide quantitative bounds of ill-known probabilities. Coherently, when information regarding some uncertain variable is given in both probabilistic and possibilistic terms, the two descriptions should be consistent. The weakest, but most intuitive consistency condition that should be respected is that an event which is probable to some degree must be possible at least to the same degree, *i.e.*,

$$p_Y(A) \leq \Pi(A) \quad \forall A \in P(Y) \tag{6.18}$$

The strongest consistency condition would require, on the other hand, that any event with nonzero probability must be fully possible, *i.e.*

$$p_Y(A) > 0 \Rightarrow \Pi(A) = 1 \quad \forall A \in P(Y) \tag{6.19}$$

The degree of probability/possibility consistency can be measured in terms of the underlying probability and possibility distributions p_Y and π_Y:

$$c(p_Y, \pi_Y) = \sum_{y \in U_Y} P_Y(y)\pi_Y(y) \tag{6.20}$$

In various applications, probability-possibility transformations are necessary, whose consistency must be assured. Several types of transformations exist, ranging from simple ratio scaling to more sophisticated operations **[Klir et Yuan 1995]**.

6.2 Uncertainty propagation

Referring to the uncertainty propagation task framed in the introduction, let a model whose output is a function $Z = f(\mathbf{Y}) = f(Y_1, Y_2, \dots, Y_j, \dots, Y_n)$ of n uncertain variables $Y_j, j = 1 \dots n$, that are "possibilistic", *i.e.*, their uncertainty is described by possibility (or fuzzy) distributions $\pi_{Y1}(y_1), \pi_{Y2}(y_2), \dots, \pi_{Yj}(y_j), \dots, \pi_{Yn}(y_n)$. In such a case, the propagation of uncertainty in possibility theory can be performed by Fuzzy Interval Analysis (FIA) **[Giles 1982; Kaufmann et Gupta 1985; Dubois et Prade 1988; Klir et Yuan 1995; Ferson 1994b]**. In summary, it can be seen the convolutions that define fuzzy arithmetic essentially reduce to interval arithmetic repeated many times, once for each level α of possibility; but, unlike interval analysis, fuzzy arithmetic yields an entire (possibility) distribution rather than a simple range.

In more detail, the operative steps of the procedure are the following **[Klir et Yuan 1995]**:

1. set α = 0;
2. select the α-cuts $A_\alpha^{Y1}, A_\alpha^{Y2}, \dots, A_\alpha^{Yj}, \dots A_\alpha^{Yn}$ of the possibility distributions $\pi_{Y_1}(y_1), \pi_{Y_2}(y_2), \dots, \pi_{Yj}(y_j), \dots, \pi_{Y_n}(y_n)$ of the "possibilistic" variables $Y_j, j = 1 \dots n$, as intervals of possible values $\left[\underline{y}_{j,\alpha}, \bar{y}_{j,\alpha}\right], j = 1 \dots n$;
3. calculate the smallest and largest values of $Z = f(\mathbf{Y}) = f(Y_1, Y_2, \dots, Y_j, \dots, Y_n)$, denoted by $\underline{z}_\alpha$ and $\bar{z}_\alpha$, respectively, letting variables Y_j range within the intervals $\left[\underline{y}_{j,\alpha}, \bar{y}_{j,\alpha}\right], j = 1, 2, \dots, n$; in particular, $\underline{z}_\alpha = \inf_{j, Y_j \in [\underline{y}_{j,\alpha}, \bar{y}_{l,\alpha}]} f(Y_1, Y_2, \dots, Y_j, \dots, Y_n)$ and $\bar{z}_\alpha = \sup_{j, Y_j \in [\underline{y}_{j,\alpha}, \bar{y}_{l,\alpha}]} (Y_1, Y_2, \dots, Y_j, \dots, Y_n)$;
4. take the values $\underline{z}_\alpha$ and $\bar{z}_\alpha$ found in step 3. above as the lower and upper limits of the α-cut A_α^Z of Z;
5. if α < 1, then set α = α + Δα (*e.g.*, Δα = 0.001) and return to step 2. above; otherwise, stop the algorithm: the possibility distribution $\pi_Z(z)$ of $Z = f(Y_1, Y_2, \dots, Y_n)$ is constructed as the collection of the values $\underline{z}_\alpha$ and $\bar{z}_\alpha$ for each α-cut (notice that since Δα = 0.001 then $N_\alpha = (q+1) = \frac{1}{\Delta\alpha} + 1 = \frac{1}{0.001} + 1 = 1001$ values of α are considered in the procedure, *i.e.*,

N_α = 1001 α-cuts of the possibility distributions $\pi_{Y_1}(y_1), \pi_{Y2}(y_2), ..., \pi_{Yj}(y_j), ..., \pi_{Yn}(y_n)$ are selected; thus, the possibility distribution $\pi_Z(z)$ of $Z = f(Y_1, Y_2, ..., Y_n)$ is constructed as the collection of its $N_\alpha = (q + 1) = 1/(\Delta\alpha) + 1 = 1/0.001 + 1 = 1001$ α-cut intervals $[\underline{z}_\alpha, \bar{z}_\alpha]$).

It is worth noting that performing an interval analysis on α-cuts assumes *total dependence* between the epistemically-uncertain parameters. Actually, this procedure implies strong dependence between the information sources (*e.g.*, the experts or observers) that supply the input possibility distributions, because the same *confidence level* (1 - α) is chosen to build the α-cuts for all the uncertain variables **[Baudrit et Dubois 2006]**.

By way of example, let Y_1 be represented by a trapezoidal possibility distribution $\pi_{Y_1}(y_1)$ with core $[c_1, d_1] = [3.5, 4]$ and support $[a_1, b_1] = [3, 5]$ (figure 6.5, top, left), and Y_2 be represented by a triangular possibility distribution $\pi_{Y_2}(y_2)$ with core $c_2 = 6$ and support $a_2, b_2] = [3.8, 7]$ (figure 6.5, top, right). Figure 6.5, bottom shows the trapezoidal possibility distribution $\pi_Z(z)$ of the output $Z = Y_1 + Y_2$ obtained by FIA with $N_\alpha = (q+1) = (1/\Delta\alpha+1) = (1/0.001+1) = 1001$ α-cut intervals. For illustration purposes, the α-cuts $A^{Y1}_{0.3} = [3.15, 4.70]$, $A^{Y2}_{0.3} = [4.46, 6.70]$ and $A^{Z}_{0.3} = [7.61, 11.40]$ of level α = 0.3 of the possibility distributions $\pi_{Y_1}(y_1)$, $\pi_{Y_2}(y_2)$ and $\pi_Z(z)$, respectively, are indicated by arrows; it is also shown in detail that $A^{Z}_{0.3} = A^{Y1}_{0.3} + A^{Y2}_{0.3} = [3.15, 4.70] + [4.46, 6.70] = [7.61, 11.40]$.

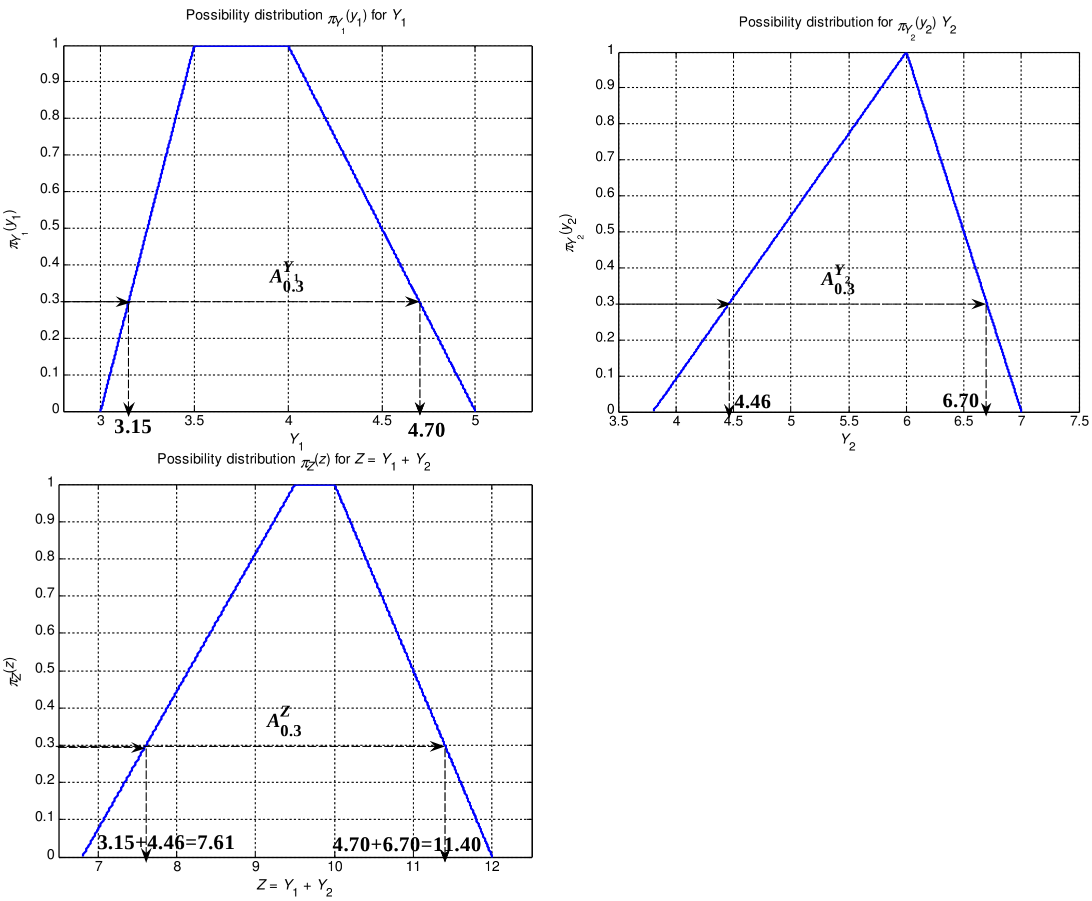

Figure 6.5 – *Top, left: trapezoidal possibility distribution $\pi_{Y_1}(y_1)$ for Y_1 with core $[c_1, d_1]$ = [3.5, 4] and support $[a_1, b_1]$ = [3, 5]; top, right: triangular possibility distribution $\pi_{Y_2}(y_2)$ for Y_2 with core c_2 = 6 and support $[a_2, b_2]$ = [3.8, 7]; bottom, left: trapezoidal possibility distribution $\pi_Z(z)$ of the output $Z = Y_1 + Y_2$ obtained by FIA with $N_\alpha = (1/\Delta\alpha + 1)$ = (1/0.001 + 1) = 1001 α-cut intervals. The α-cuts $A^{Y1}_{0.3}$ = [3.15, 4.70], $A^{Y2}_{0.3}$ = [4.46, 6.70] and $A^{Z}_{0.3}$ = [7.61, 11.40] of level α = 0.3 of the possibility distributions $\pi_{Y_1}(y_1)$, $\pi_{Y_2}(y_2)$ and $\pi_Z(z)$, respectively, are also indicated by arrows.*

6.3 Discussion

Based on the considerations and examples above, the **advantages of possibility theory** (fuzzy interval analysis) can be summarized as follows **[Ferson 1994b, 1999]**:

▷ computations are *simple* and *easy* to explain. Because possibility distributions (fuzzy numbers) have a simple structure, the convolutions that define their arithmetic essentially reduce to *interval arithmetic* repeated *many times*, once *for each level of possibility* **[Baudrit et Dubois 2006]**. But, unlike interval analysis, fuzzy arithmetic yields an entire distribution rather than a simple range;

▷ possibility distributions (fuzzy numbers) can be assigned *subjectively*;

▷ possibility theory does *not* require *detailed empirical information*;

▷ possibility theory works with *non-statistical* (*i.e.*, non-probabilistic) uncertainty; thus, it is applicable to *all kinds of uncertainty*;

▷ possibility theory needs *fewer* arbitrary (and often unjustified) *assumptions* than probability theory;

▷ possibility distributions (fuzzy numbers) are a *generalization* and refinement of *intervals* in which the bounds vary according to the level of confidence one has in the estimation;

▷ possibility theory is *intermediate* in *conservatism* between analogous Monte Carlo and interval approaches. Actually, it allows analysts to construct a much more highly resolved picture of risks of various magnitudes than interval analysis does, but it does so without being as information-intensive (or assumption-intensive) as Monte Carlo methods generally have to be. An often cited example of how probability theory, possibility theory and interval analysis differ compares the sums of uniform distributions (figure 6.6). When two uniform probability distributions $p_Y(y) = U[0,1)$ (figure 6.6, top, left) are added together under the assumption of independence, the resulting sum $Z = (Y + Y)$ is distributed according to a triangular distribution $p_{Z=Y+Y}(z) = TR(0,1,2)$ (figure 6.6, middle, left). In the limit, the sum $Z = Y + Y + \ldots + Y$ tends to a normal distribution (with a very small coefficient of variation) (figure 6.6, bottom, left shows the probability distribution $p_{Z=Y+Y+\ldots+Y}(z)$ of the sum $Z = Y + Y + \ldots + Y$ of ten uniform probability distributions $p_Y(y) = U[0,1)$). When two analogous flat fuzzy numbers (*i.e.*, intervals) are added together (solid line in figure 6.6, top, right), the result is another flat distribution (solid line in figure 6.6, middle, right), and in the limit, still a flat distribution (solid line in figure 6.6, bottom, right). The big difference here is that fuzzy arithmetic is *not* assuming *independence* between the variables. Of course, when the input distributions are not flat (*e.g.*, see dashed line in figure 6.6, top, right), the answer coming out of fuzzy arithmetic will not be either, but the distribution will be broader than that predicted by the comparable probabilistic method (*e.g.*, compare the solid line in figure 6.6, bottom, left with the dashed line in figure 6.6, bottom, right). However, it won't be as broad or hyper-conservative as the analogous interval analysis approach (*e.g.*, compare the solid and dashed lines in figure 6.6, bottom, right);

▷ possibility distributions (fuzzy numbers) *maintains conservatism* under uncertainty about dependencies among variables;

▷ although fairly simple, possibility distributions (fuzzy numbers) are very *robust* representations when empirical information is very sparse. In other words, there is only *weak sensitivity* of the final results to details of the *shapes* of the input possibility distributions. Many analysts consider this an important advantage because there is often so little dependable empirical information underlying the selection of one input distribution over many alternatives.

On the contrary, the **disadvantages of possibility theory** (fuzzy interval analysis) can be summarized as follows **[Ferson 1994b, 1999]**:

▷ it is *not* yet *widely known and applied* so that it has not been broadly accepted in the risk assessment community. Much effort has been made in this area, often with a mathematical orientation, but no convincing framework for risk assessment in practice presently exists based on these alternative theories. Further research is required to make these alternatives operational in a risk assessment context;

- it may be *overly conservative*;
- *repeated parameters* may represent a computational problem (as for interval analysis);
- it is not clear if it is correct to *merge* numbers whose *conservatisms* are *different*; in other words, are alpha levels comparable for different variables?
- in the uncertainty propagation by fuzzy interval analysis, it implicitly assumes *total* (*i.e., perfect*) *dependence* between the uncertain variables.

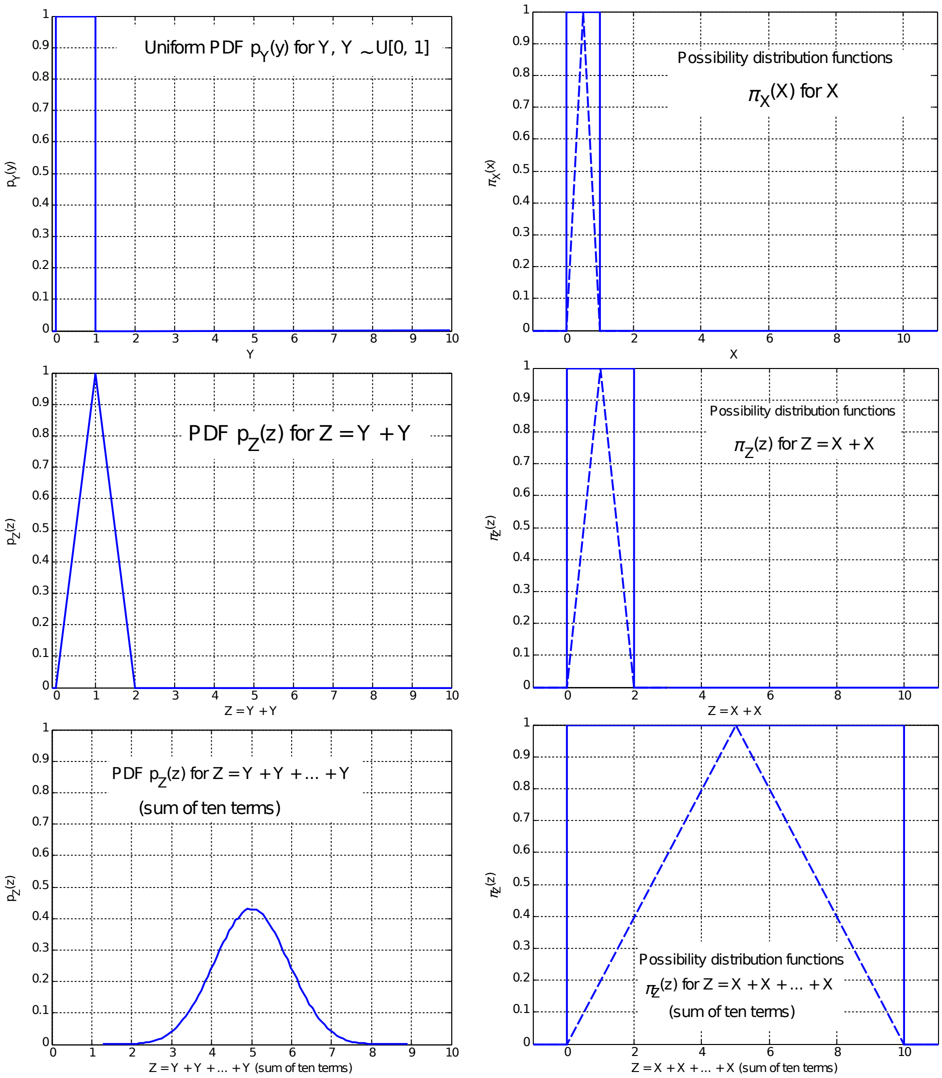

Figure 6.6 – *Top, left: uniform probability distributions* $p_Y(y) = U[0,1)$ *for uncertain variable* Y*; top, right: interval [0, 1] (solid line) and triangular possibility distribution* $\pi_X(x) = TR(0, 0.5, 1)$ *(dashed line) for uncertain variable* X*; middle, left: triangular probability distribution* $p_Z(z) = TR(0,1,2)$ *for* $Z = Y + Y$*; middle, right: interval [0, 2] (solid line) and triangular possibility distribution* $\pi_Z(z) = TR(0,1,2)$ *(dashed line) for* $Z = X + X$*; bottom, left: probability distribution* $p_Z(z)$ *for* $Z = \sum_{i=1}^{10} Y_i Y_i \sim U[0,1)$*; bottom, right: interval [0, 10] (solid line) and triangular possibility distribution* $\pi_Z(z) = TR(0,5,10)$ *(dashed line) for* $Z = \sum_{i=}^{10} X_i, X_i \sim \pi_{X_i}(x_i) = TR(0, 0.5, 1)$

Finally, it is worth noting that important work has also been carried out to combine different approaches, for example probabilistic analysis and possibility theory. Here the uncertainties of some parameters are represented by probability distributions and those of some other parameters by means of possibilistic distributions. An integrated computational framework has been proposed for jointly propagating the probabilistic and possibilistic uncertainties [Baudrit et Dubois 2005, 2006; Baudrit et al. 2007a,b, 2008; Cooper et al. 1996; Guyonnet et al. 2003; Kentel et Aral 2004, 2005, 2007; Möller 2004; Möller et Beer 2004, 2008; Möller et al. 2003, 2006]. This framework has previously been tailored to event tree analysis [Baraldi et Zio 2008] and fault tree analysis [Flage et al. 2010a], allowing for the uncertainties about event probabilities (chances) to be represented and propagated using both probability and possibility distributions. The work has been extended in [Flage et al. 2010b, 2013] by comparing the results of the hybrid approach with those obtained by purely probabilistic and possibilistic approaches, using different probability/possibility transformations.

7

Concerns for practical decision-making

From the front end of the analysis, the representation of the knowledge available as input to the risk assessment in support of the decision-making must be **faithful and transparent**: the methods and models used should not add information that is not there, nor ignore information that is there. In high-consequence technologies, one deals with rare events and processes for which experimental and field data are lacking or scarce, at best; then it is essential that the related information and knowledge are elicited and treated in an adequate way. Two concerns then need to be balanced:

- $\boxed{\text{C1}}$ the knowledge should to the extent possible be "inter-subjective" in the sense that the representation corresponds to documented and approved information and knowledge;
- $\boxed{\text{C2}}$ the analysts' judgments ('degrees of belief') should be clearly reflected.

The former concern makes the pure Bayesian approach difficult to apply: introducing analysts' subjective probability distributions is unjustifiable since this leads to building a structure in the probabilistic analysis that is not present in the "approved" expert-provided information. For example, if an expert states his or her uncertainty assessment on a parameter value in terms of a range of possible values, this does not justify the allocation of a specific distribution function (for example the uniform distribution) onto the range. In this view, it might be said that a more defense-in-depth (bounding) representation of the information and knowledge available would be one which leaves the analysis open to all possible probability distribution structures on the assessed range, without imposing one in particular and without excluding any, thus providing results which bound all possible distributions.

On the other hand, the representation framework should also take into account the concern $\boxed{\text{C2}}$, *i.e.* allow for the transparent inclusion of preferential assignments by the experts (analysts) who wish to express that some values are more or less likely than others. The Bayesian approach is the proper framework for such assignments.

From the point of view of the quantitative modeling of uncertainty in risk assessment, two topical issues are the proper handling of dependencies among uncertain parameters, and of model uncertainties. No matter what modeling paradigm is adopted, it is critical that the meaning of the various concepts be clarified. Without such clarifications it is impossible to build a scientific-based risk assessment. In complex situations, when the propagation is based on many parameters, strong assumptions may be required to be able to carry out the analysis. The analysts may acknowledge a degree of dependency, but the analysis may not be able to describe it in an adequate way. The derived uncertainty representations must be understood and communicated as measures conditional on this constraint. In practice, it is a main task of the analysts to seek simple representations of the system performance and by "smart" modelling it is often possible to "obtain" independence. The models used are also included in the background knowledge of epistemic-based uncertainty representations. We seek accurate models, but at the same time simple models. The choice of the right model cannot be seen in isolation from the purpose of the risk assessments.

From the back-end of the analysis, *i.e.* the use of its outcomes for practical decision-making, it is fundamental that the meaning and practical interpretation of the quantities computed are communicated in an understandable format to the decision-makers. The format must allow for meaningful comparisons with numerical safety criteria if defined, for manipulation (*e.g.* by screening, bounding and/or sensitivity analyses) and for transformation in deliberation processes.

The context of risk and uncertainty assessments must be seen within and for decision-making. There are in fact also different perspectives on how to *use* risk and uncertainty assessments for decision-making. Strict adherence to expected utility theory, cost-benefit analysis and related theories would mean clear recommendations on what is the optimal arrangement or measure. However, most practical oriented analysts would see risk and uncertainty assessments as decision support tools, in the sense that the assessments inform the decision-makers. The decision-making is risk-informed, not risk-based **[Apostolakis 2004]**. In general, there is a significant leap from the decision-making basis to the decision. What this leap (often referred to as managerial review and judgment) comprises is a subject being discussed in the literature (*e.g.* **[Aven 2010a]**) and it is also closely linked to the present work. The scope and boundaries of risk and uncertainty assessments define to a large extent the content of this review and judgment. A narrow risk and uncertainty characterization calls for a broader managerial review and judgment, and *vice versa.*

Seeing risk assessment as an aid for decision-making, alternative approaches for the representation and treatment of uncertainties in risk assessment are required. Different approaches provide a broader and more informative decision basis than one approach alone. A Bayesian analysis without thorough considerations of the background knowledge and associated assumptions would normally fail to reveal important uncertainty factors. Such considerations (qualitative assessments) are essential for ensuring that the decision-makers are not seriously misled by the risk assessment results.

It is a huge step from such assessments to methods that quantitatively express, and bound, the imprecision in the probability assignments. These methods are also based on a set of premises and assumptions, but not to the same degree as the pure probability-based analyses. Their motivation is that the intervals produced correspond better to the information available. For example, an analysis (*e.g.*, based on possibility theory) results in an interval $[a, b]$ for the subjective probability $P(A)$ of event A. The analysts are not able or willing to precisely assign their probability $P(A)$. The decision-maker may however request that the analysts make such assignments — the decision-maker would like to be informed by the analysts' degree of belief (refer to concern $\boxed{\text{C2}}$ above). The analysts are consulted as experts in the field studied and the decision-maker expects them to give their faithful report of the epistemic uncertainties about the unknown quantities addressed. The decision-maker knows that these judgments are based on some knowledge and some assumptions, and are subjective in the sense that others could conclude differently, but these judgments are still considered valuable as the analysts (and the experts they use in the analysis) have competence in the field being studied. The analysts are trained in probability assignments and the decision-maker expects that the analysts are able to transform their knowledge into probability figures **[Aven 2010b]**.

Following this view, we should continue to conduct probability–based analysis reflecting the analysts' degree of belief about unknown quantities, but we should also encourage additional assessments. These include sensitivity analyses to see how sensitive the risk indices are with respect to changes in basic input quantities, for example assumptions and suppositions **[Helton et al. 2006; Cacuci et Ionescu-Bujor 2004; Saltelli et al. 2008; Frey et Patil 2002]**, but also crude qualitative assessments of uncertainty factors, as mentioned above. The use of imprecision intervals would further point at the importance of key assumptions made.

8

Discussion and conclusions

Nowadays, the use of risk assessment as a tool in support of decision-making is quite widespread, particularly in high-consequence technologies. The techniques of analysis sustaining the assessment must be capable of building the level of confidence in the results required for taking the decision they inform. A systematic and rational control on the uncertainty affecting the analysis is the key to confidence building.

In practical risk assessments, the uncertainty is commonly treated by probabilistic methods, in their Bayesian, subjective formulation for the treatment of rare events and poorly known processes typical of high-consequence technologies. However, a number of theoretical and practical challenges seem to be still somewhat open. This has sparked the emergence of a number of alternative approaches, which have been here considered in relation to the support to decision-making that they can provide.

Many researchers and analysts are skeptical of the use of "non-probabilistic" approaches (such as described in chapters 3 to 6) for the representation and treatment of uncertainty in risk assessment for decision-making. An imprecise probability result is considered to provide a more complicated representation of uncertainty **[Lindley 2000]**. By an argument that the simple should be favoured over the complicated, **[Lindley 2000]** takes the position that the complication of imprecise probabilities seems unnecessary. In a more rejecting statement, **[Lindley 2006]** argues that the use of interval probabilities goes against the idea of simplicity, as well as confuses the concept of measurement (*interpretation*, in the view of **[Bedford et Cooke 2001]**) with the practice of measurement (*measurement procedures* in the view of **[Bedford et Cooke 2001]**). The standard, **[Lindley 2006]** emphasizes, is a conceptual comparison. It provides a norm, and measurement problems may make the assessor unable to behave according to it. **[Bernardo et Smith 2000, p. 32]** call the idea of a formal incorporation of imprecision into the axiom system "an unnecessary confusion of the *prescriptive* and the *descriptive*" for many applications, and point out that measurement imprecision occurs in any scientific discourse in which measurements are taken. They make a parallel to the inherent limits of a physical measuring instrument, where it may only be possible to conclude that a reading is in the range 3.126 to 3.135, say. Then, we would typically report the value 3.13 and proceed as if this were the precise number:

> *We formulate the theory on the prescriptive assumption that we aspire to exact measurement […], whilst acknowledging that, in practice, we have to make do with the best level of precision currently available (or devote some resources to improving our measuring instruments!)* **[Bernardo et Smith 2000, p. 32]**

Many analysts argue fiercely for a strict Bayesian analysis. A typical statement is **[North 2010]**:

> *For me, the introduction of alternatives such as interval analysis to standard probability theory seems a step in the wrong direction, and I am not yet persuaded it is a useful area even for theoretical research. I believe risk analysts will be better off using standard probability theory than trying out alternatives that are harder to understand, and which will not be logically consistent if they are not equivalent to standard probability theory.*

However, as argued in this document, this approach does not solve the problems raised. The decision basis cannot be restricted to subjective probabilities: there is a need to go beyond the Bayesian approach.

In the end, any method of uncertainty representation and analysis in risk assessment must address a number of very practical questions before being applicable in support to decision-making:

- ▷ How completely and faithfully does it represent the knowledge and information available?
- ▷ How **costly** is the analysis?
- ▷ How much **confidence** does the decision-maker gain from the analysis and the presentation of the results? (This opens up the issue of how one can measure such confidence).
- ▷ What value does it bring to the dynamics of the **deliberation process**?

Any method which intends to complement, or in some justified cases supplement, the commonly adopted probabilistic approach to risk assessment should demonstrate that the efforts needed for the implementation and familiarization, by the analysts and decision-makers, are feasible and acceptable in view of the benefits gained in terms of the above questions and, eventually, of the confidence in the decision made.

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